Hyperbolic triangle chains and finite mapping class group orbits

j.w. Samuel Bronstein

Arnaud Maret
Strasbourg University

Painlevé Seminar
14/05/2025

algebraic solutions to isomonodromy differential equations (Painlevé VI, Schlesinger systems)

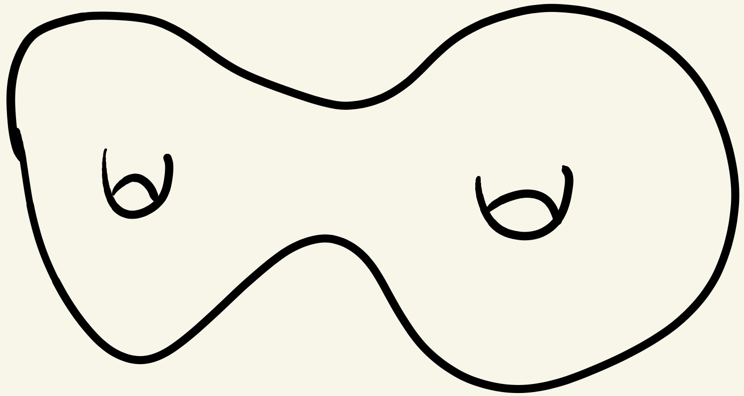
$\rightsquigarrow$

finite mapping class group orbits on $SL_2\mathbb{C}$ character varieties

(1) Set-up and history

(2) Constructing / classifying finite orbits

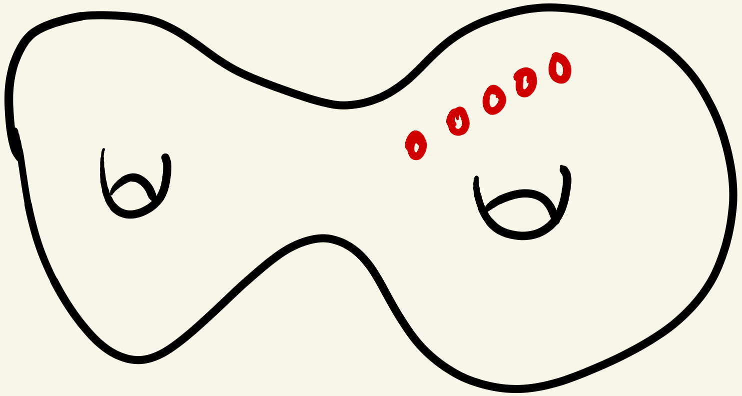
(1) Set-up and history

$S =$  : oriented surface $g \geq 0$

mapping class group of S : $\mathrm{Mod}(S)$

$$\pi_0(\mathrm{Homeo}_+(S)) \cong \mathrm{Out}(\pi_1(S)) \cong \pi_1 \mathcal{M}_g$$

(topological) (algebraic) (geometric)

$S =$  : oriented surface

$g \geq 0$, $n \geq 0$ punctures

pure mapping class group of S : $\mathrm{PMod}(S)$

$$\pi_0\left(\mathrm{Homeo}^*_+(S)\right) \cong \mathrm{Out}^*(\pi_1(S)) \cong \pi_1 \mathcal{M}_{g,n}$$

(topological) (algebraic) (geometric)

G : Lie / algebraic group

character variety of (S,G) : $\mathrm{Rep}(S,G) = \mathrm{Hom}(\pi_1 S, G) \big/ G$

(topological quotient)

$$\mathcal{P}\mathrm{Mod}(S) \hookrightarrow \mathrm{Rep}(S,G)$$

(by precomposition)

goal: understand finite orbits of

$$\mathrm{PMod}(S) \curvearrowright \mathrm{Rep}(S, SL_2\mathbb{C})$$

example 0:

$$\rho : \pi_1 S \longrightarrow SL_2\mathbb{C} \ : \ \text{finite image}$$

$$\Rightarrow [\rho] \in \mathrm{Rep}(S, SL_2\mathbb{C}) \text{ has finite orbit}$$

$g \geq 1, \; n \geq 0$

$$\boxed{\begin{array}{l} [\rho] \in \mathrm{Rep}(S, SL_2\mathbb{C}) \text{ has finite orbit} \\ \quad \overset{*}{\Longrightarrow} \rho \text{ has finite image} \end{array}}$$

* not completely true for $g = 1$ (infinite dihedral group)

[Biswas – Gupta – Mj – Whang 22', Cousin – Heu 21']

$\underline{g=0}$

* $n=3$: PMod(S) is trivial

  $\Rightarrow$ every $[\rho]$ is a finite orbit

* $n=4$: $\boxed{\text{algebraic solutions to Painlevé VI} \leftrightsquigarrow \text{finite orbits in } \mathrm{Rep}(S, SL_2\mathbb{C})}$

examples : Fuchs, Hitchin, Dubrovin, Mazzocco, Andreev, Kitaev, Boalch, ...

full list : Boalch '06 + Lisovyy–Tykhyy '14

* $n \geq 5$:

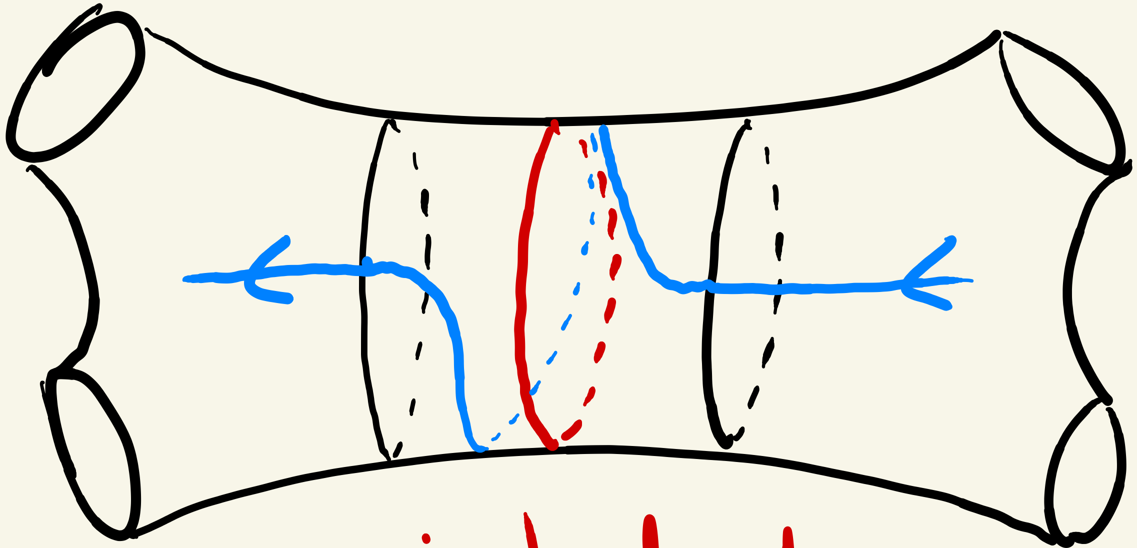
examples: Diana, Calligaris, Mazzocco, Tykhyy, ...

$$\boxed{\begin{array}{l}\underline{\text{Tykhyy's Conjecture}}\\ * \ n=5,6: \text{ list of finite orbits}\\ * \ n \geq 7: \ \nexists \text{ "new" finite orbits}\end{array}}$$

<u>Theorem</u> (Bronstein-M. 24', Lam-Landesman-Litt 23')
Tykhyy's Conjecture is true

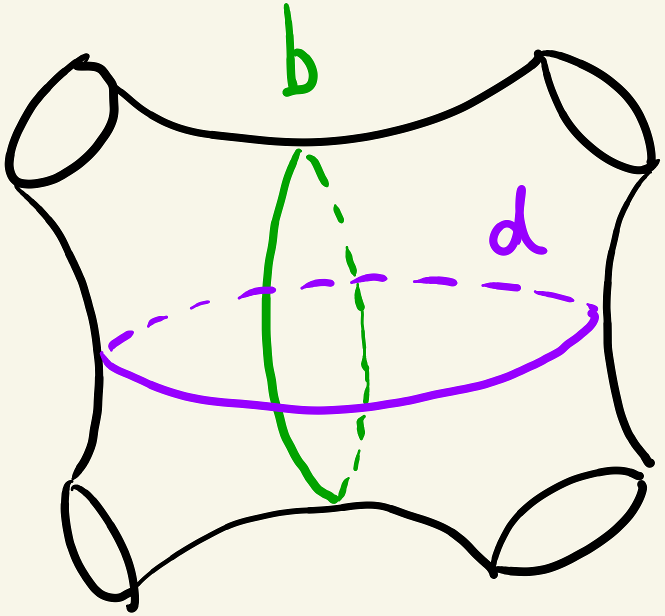
(2) Constructing / classifying finite orbits

$$\mathrm{PMod}(S) = \langle \text{Dehn twists} \rangle$$



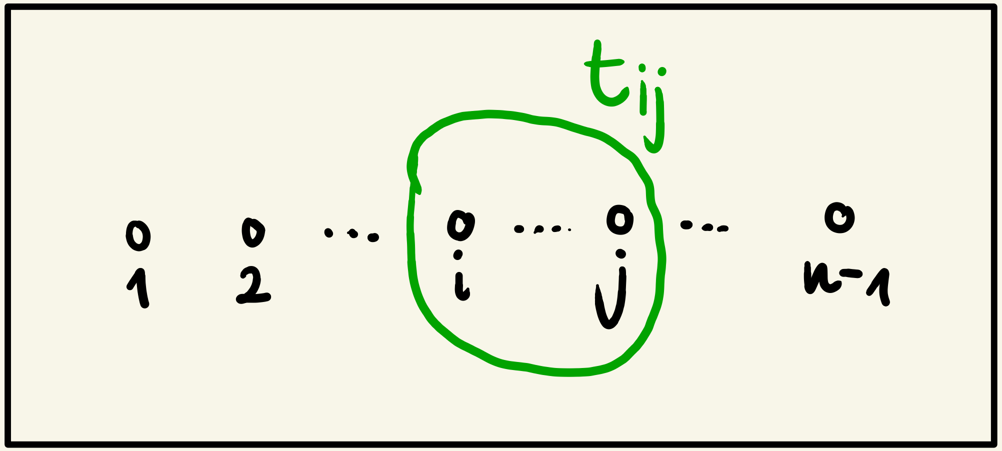
simple closed curve a

$\underline{n=4}$



$$\mathrm{PMod}(S) = \langle \tau_b, \tau_d \rangle$$

$\underline{\text{general}}$



$$\mathrm{PMod}(S) = \langle \tau_{ij} \mid \substack{i<j \\ (i,j) \neq (1,n-1)} \rangle$$

[Ghaswala - Winarski 17']

Heuristic

$[\rho] \in \mathrm{Rep}(S, SL_2\mathbb{C})$
finite orbit

$\mathrm{Tr}(\rho(a)) \in (-2,2)$

" $\Rightarrow$ " $\rho(a)$ is elliptic
$\forall a$ simple closed curve
i.e. ρ is totally elliptic

" $\Rightarrow$ " ρ is real
i.e. $\rho : \pi_1 S \to SU(2),\ SL_2\mathbb{R}$

[Procesi, Goldman-Xia]

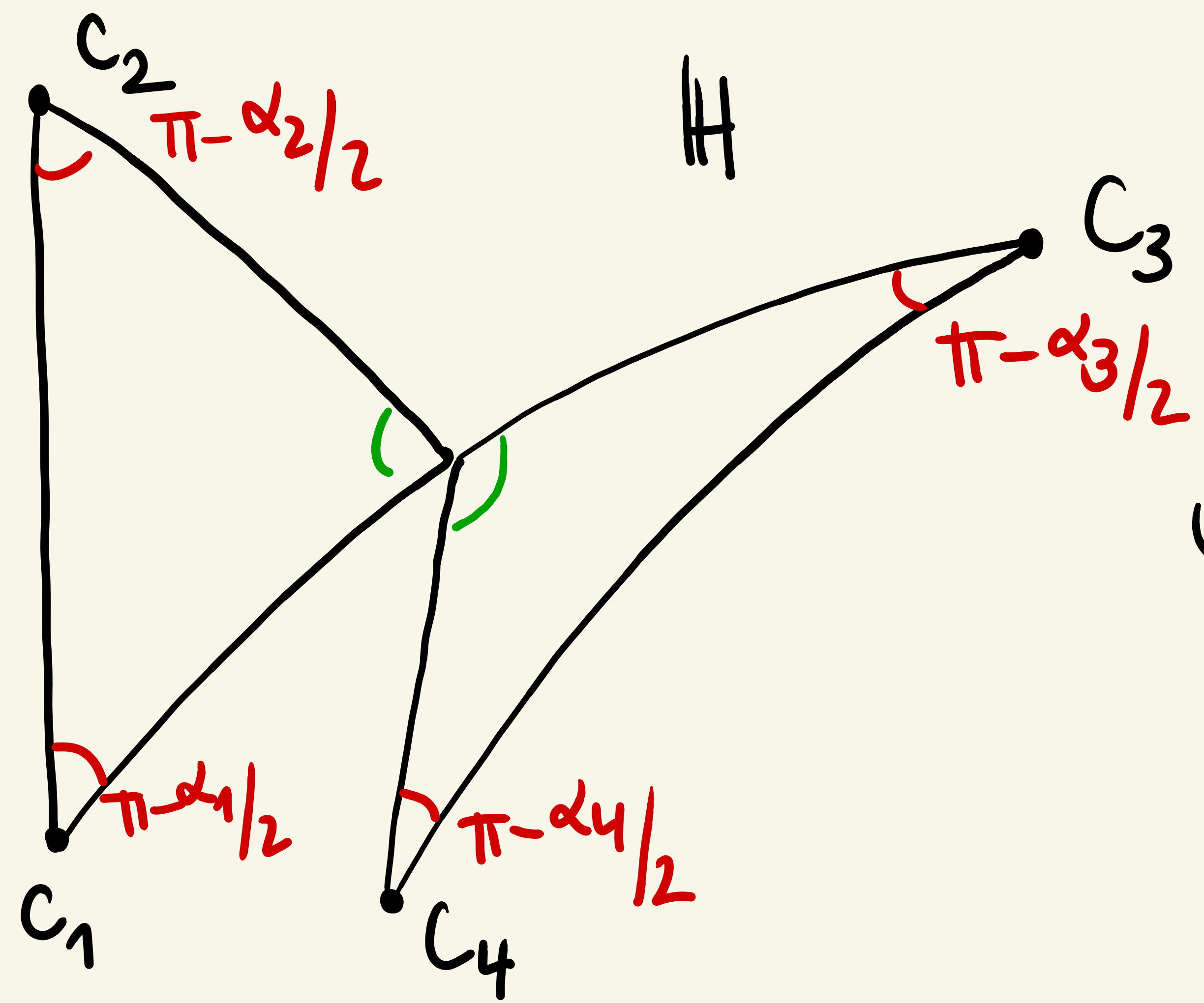
<u>moral</u> : to find finite orbits, look for ==totally elliptic==

$$\rho : \pi_1 S \longrightarrow SL_2\mathbb{R}$$

Triangle chains (M.22')

$\alpha = (\alpha_1, \dots, \alpha_4) \in (0, 2\pi)^4$

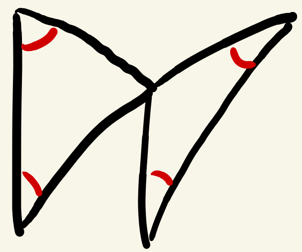
$\alpha_1 + \dots + \alpha_4 > 6\pi$

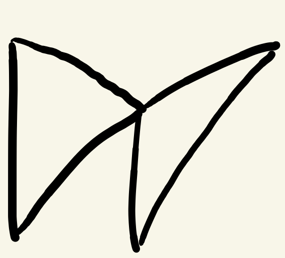
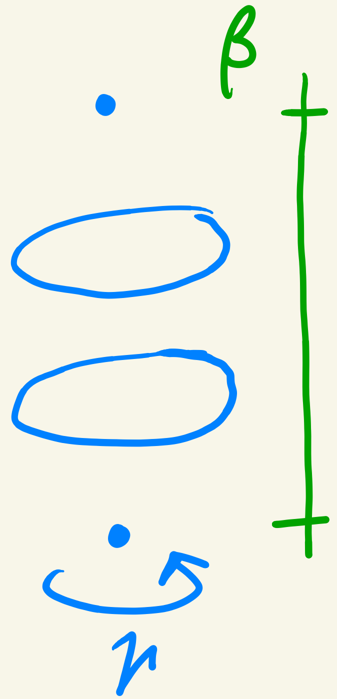


<u>conditions</u>:

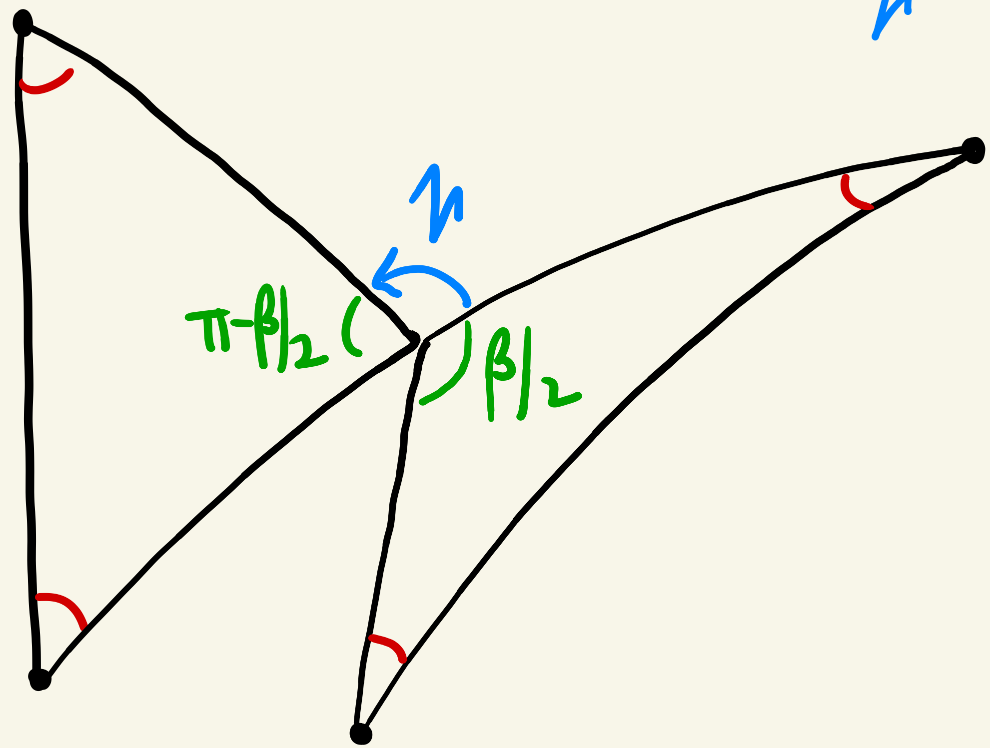
(1) $\measuredangle c_i = \pi - \alpha_i/2$

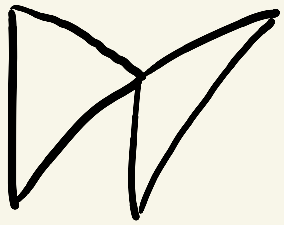
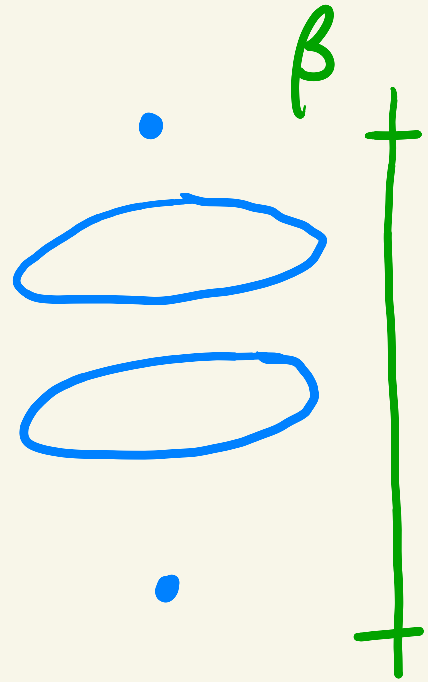
(2) $\measuredangle + \measuredangle = \pi$

$TC_{\alpha}^{n=4} := \Big\{$  $\Big\} \Big/ PSL_2\mathbb{R} = ?$

$TC^{n=4}_{\alpha} := \{$  $\} / PSL_2\mathbb{R} \cong$ 

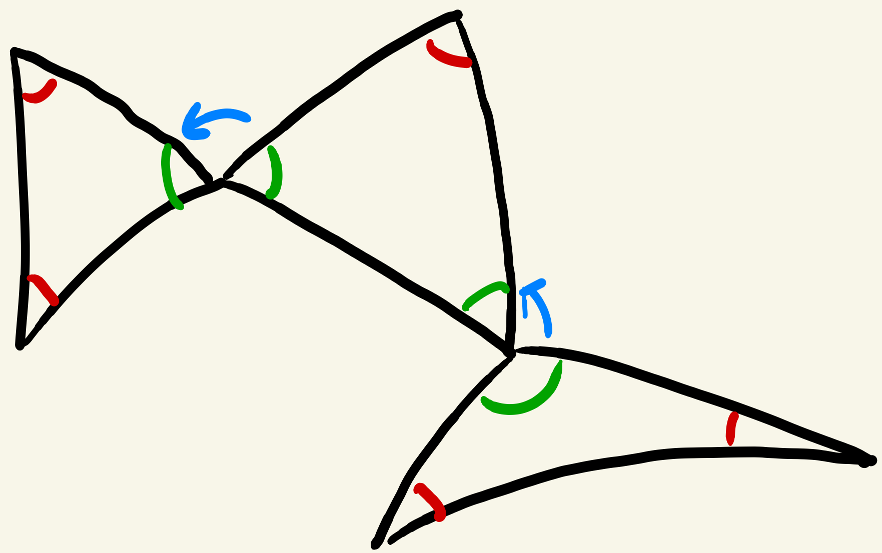
$\cong \mathbb{CP}^1$

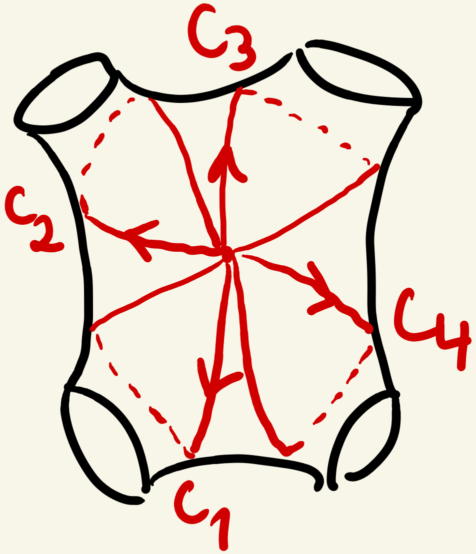


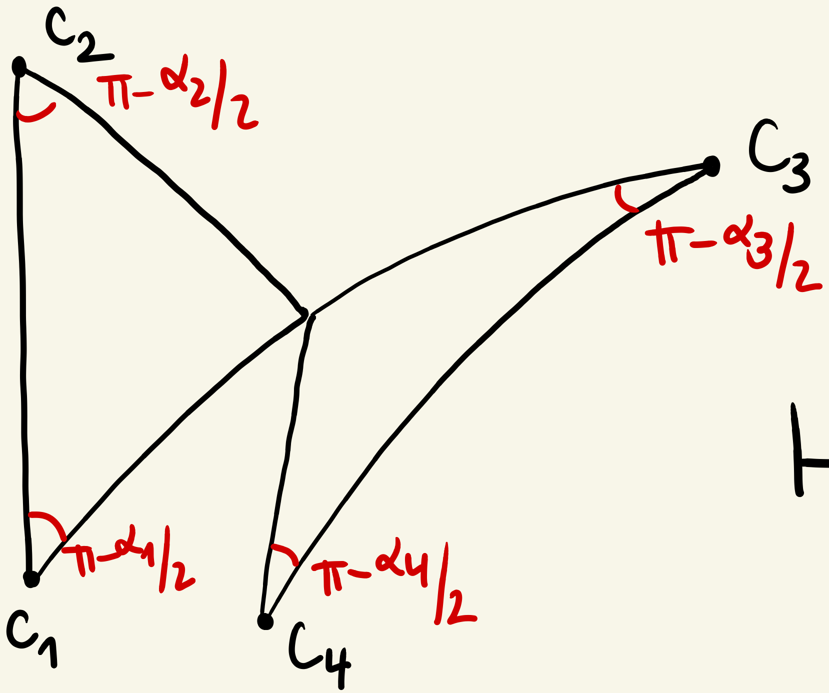
$TC_\alpha^{n=4} := \Big\{$  $\Big\} \big/ PSL_2\mathbb{R} \cong$ 

$\cong \mathbb{CP}^1$

more generally: $TC_\alpha^n \cong \mathbb{CP}^{n-3}$



$TC_{\alpha}^{n=4} \longrightarrow \mathrm{Rep}\Big($  $, PSL_2\mathbb{R}\Big)$



$\longmapsto$

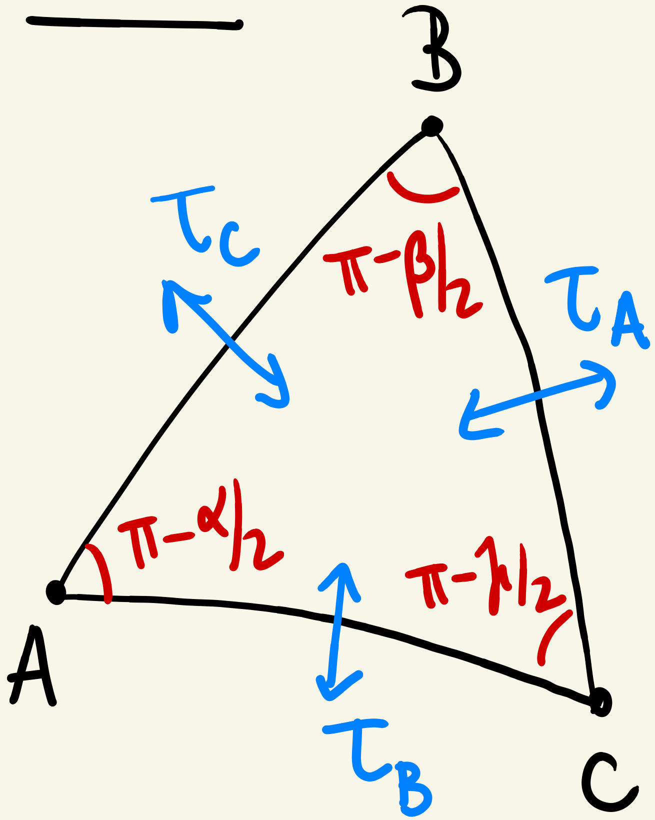
$$\langle c_1, \dots, c_4 \mid \Pi c_i = 1 \rangle$$

$$\|$$

$$\rho : \pi_1 S \longrightarrow PSL_2\mathbb{R}$$

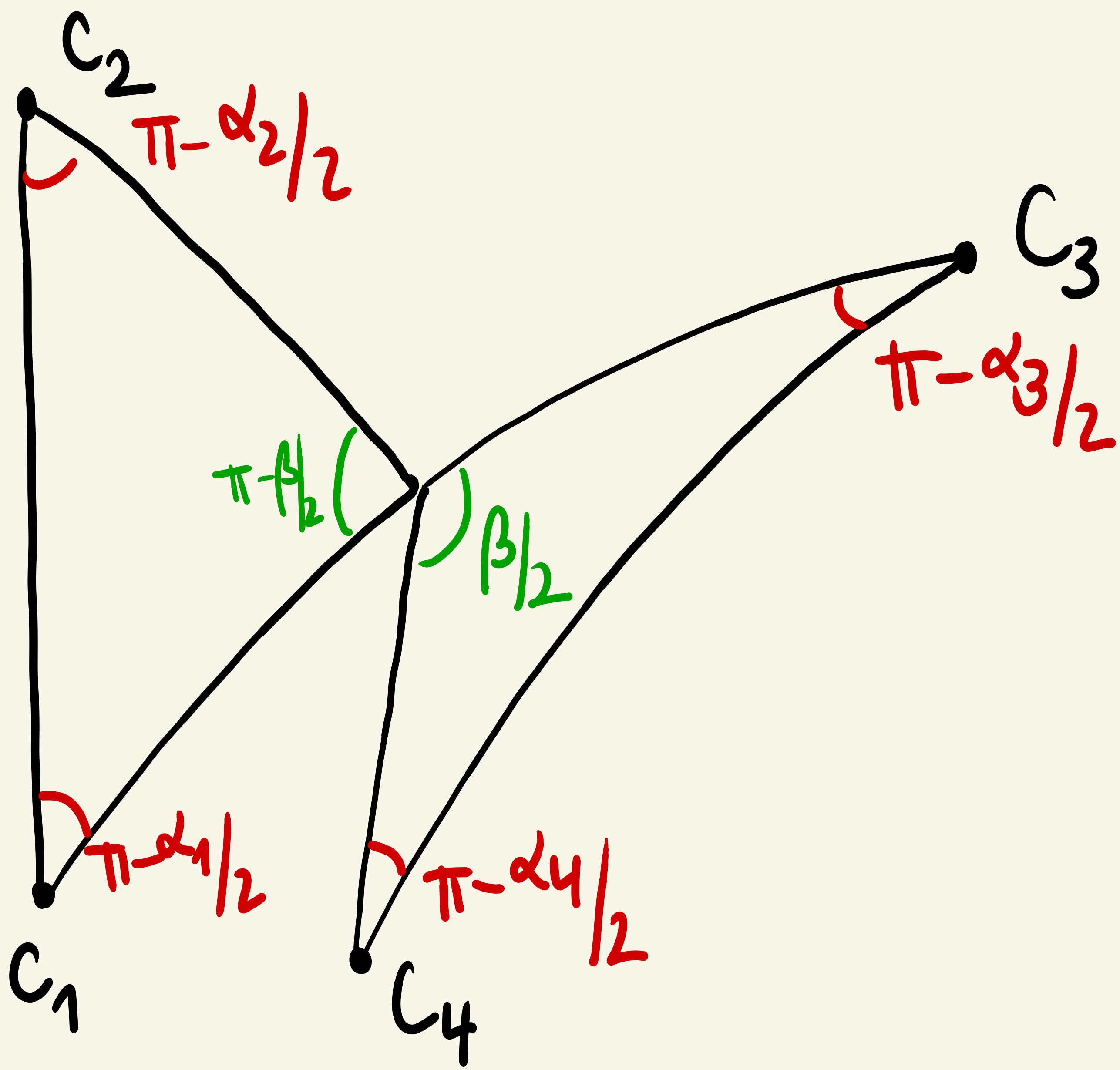
$$c_i \longmapsto \mathrm{rot}_{\alpha_i}(c_i)$$

lemma



$$\Rightarrow \underbrace{\mathrm{rot}_\alpha(A)}_{\tau_B\tau_C} \cdot \underbrace{\mathrm{rot}_\beta(B)}_{\tau_C\tau_A} \cdot \underbrace{\mathrm{rot}_\gamma(C)}_{\tau_A\tau_B} = 1$$

$\square$



c_2
$\pi - \alpha_2/2$
c_3
$\pi - \alpha_3/2$
$\pi - \beta/2$
$\beta/2$
c_1
$\pi - \alpha_1/2$
c_4
$\pi - \alpha_4/2$

$g=0, n\geq 4$

$$\mathrm{Rep}(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}(S, PSL_2\mathbb{R})$$

$\rho(c_i)$ is elliptic of angle α_i

character variety

$g=0,\ n\geq 4$

$$\mathrm{Rep}(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}(S, PSL_2\mathbb{R})$$

character variety

α- relative character variety

$g=0, n\geq 4$

$$\mathrm{Rep}(S, \mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}_\alpha(S, \mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}_\alpha^{DT}$$

character variety

α- relative character variety [Deroin-Tholozan 19']

$\exists$ compact connected component

DT component

$g=0,\ n\geq 4$

$$\begin{array}{ccccc}
\mathrm{Rep}(S,\mathrm{PSL}_2\mathbb{R}) & \supseteq & \mathrm{Rep}_\alpha(S,\mathrm{PSL}_2\mathbb{R}) & \supseteq & \mathrm{Rep}^{DT}_\alpha \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
\mathrm{PMod}(S) & & \mathrm{PMod}(S) & & \mathrm{PMod}(S)
\end{array}$$

$g=0,\ n\geq 4$

(M.24')

$$\mathrm{Rep}(S, \mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}(S, \mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}^{DT} \cong TC_{\alpha}$$

$$\circlearrowleft \qquad\qquad\qquad \circlearrowleft \qquad\qquad\qquad \circlearrowleft$$

$$\mathrm{PMod}(S) \qquad\qquad \mathrm{PMod}(S) \qquad\qquad \mathrm{PMod}(S)$$

character variety

α-relative character variety

DT component

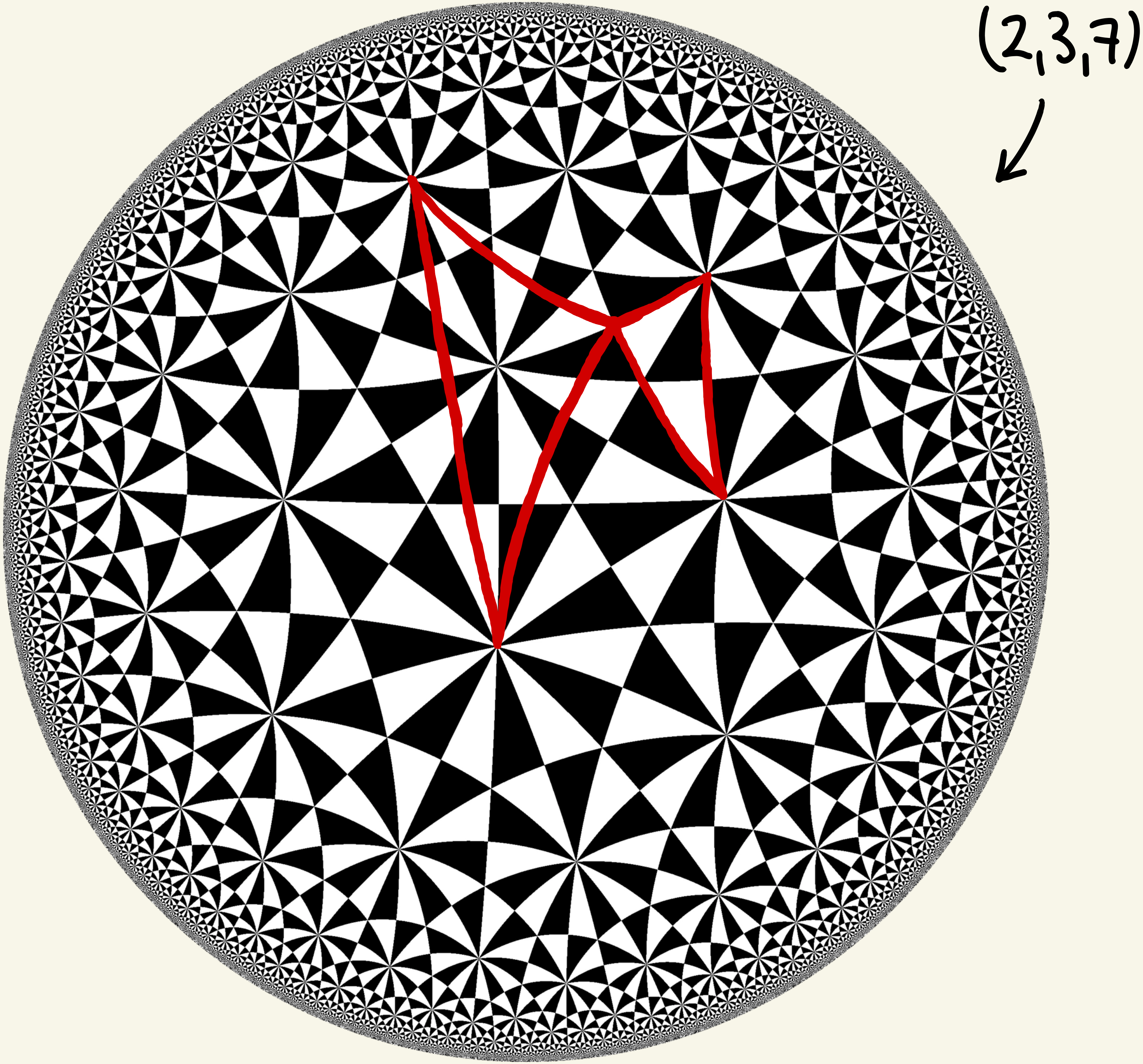
totally elliptic!

exemple 1

$[\rho] \in \mathrm{Rep}^{DT}_{\alpha}$ + ρ has discrete image

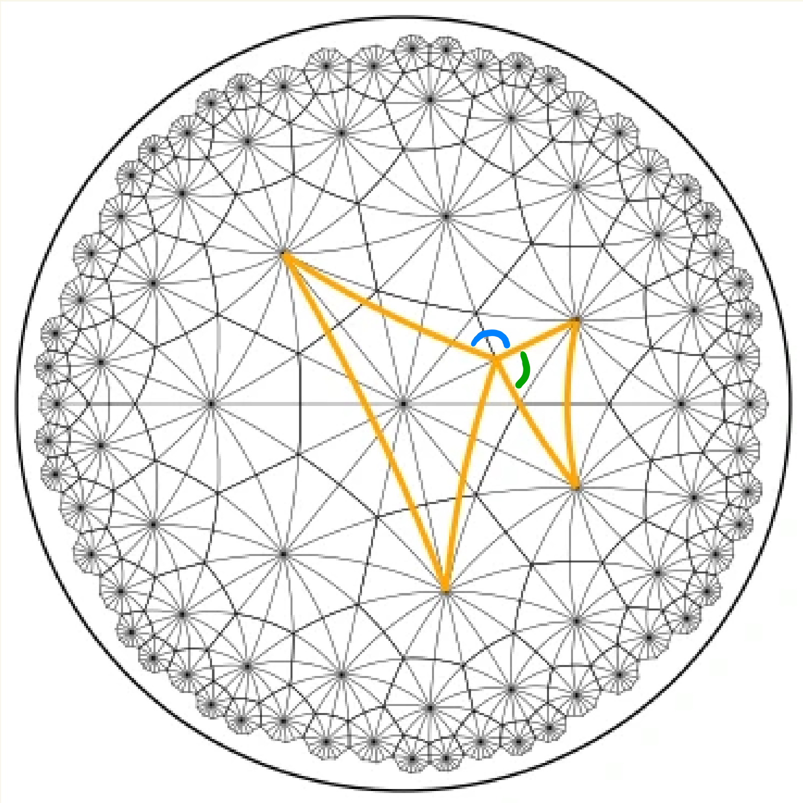
$\Rightarrow$ orbit of $[\rho]$ is discrete in $\mathrm{Rep}^{DT}_{\alpha}$

$\Rightarrow$ orbit of $[\rho]$ is finite!

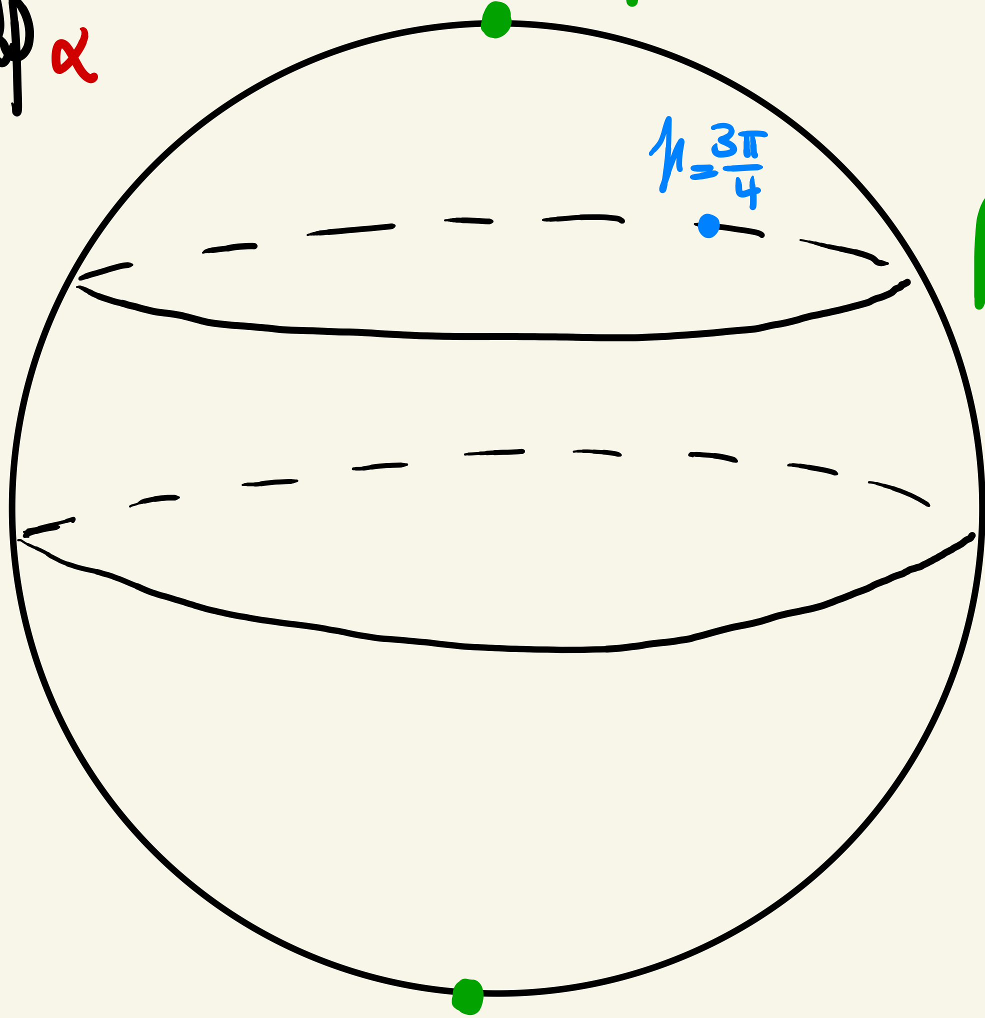

(2,3,7)

$\mathrm{Rep}^{DT}_{\alpha}$

$\beta = \frac{8\pi}{7}$



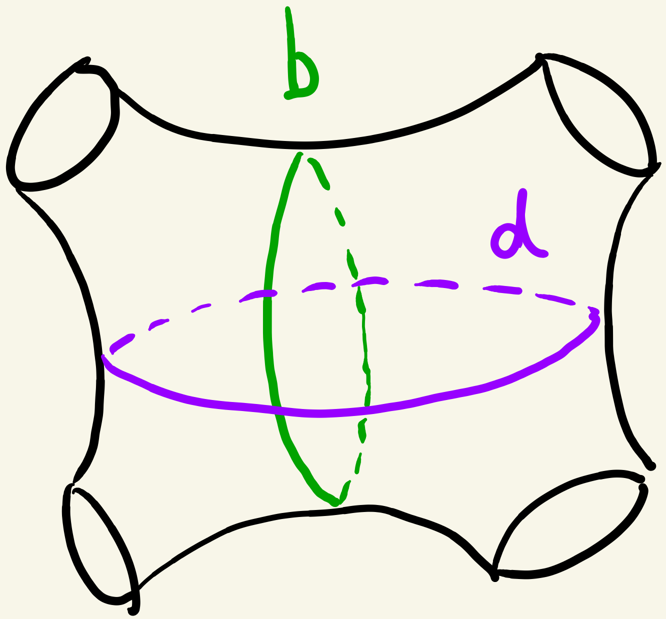
$\beta = \pi \quad \gamma = \frac{3\pi}{4}$



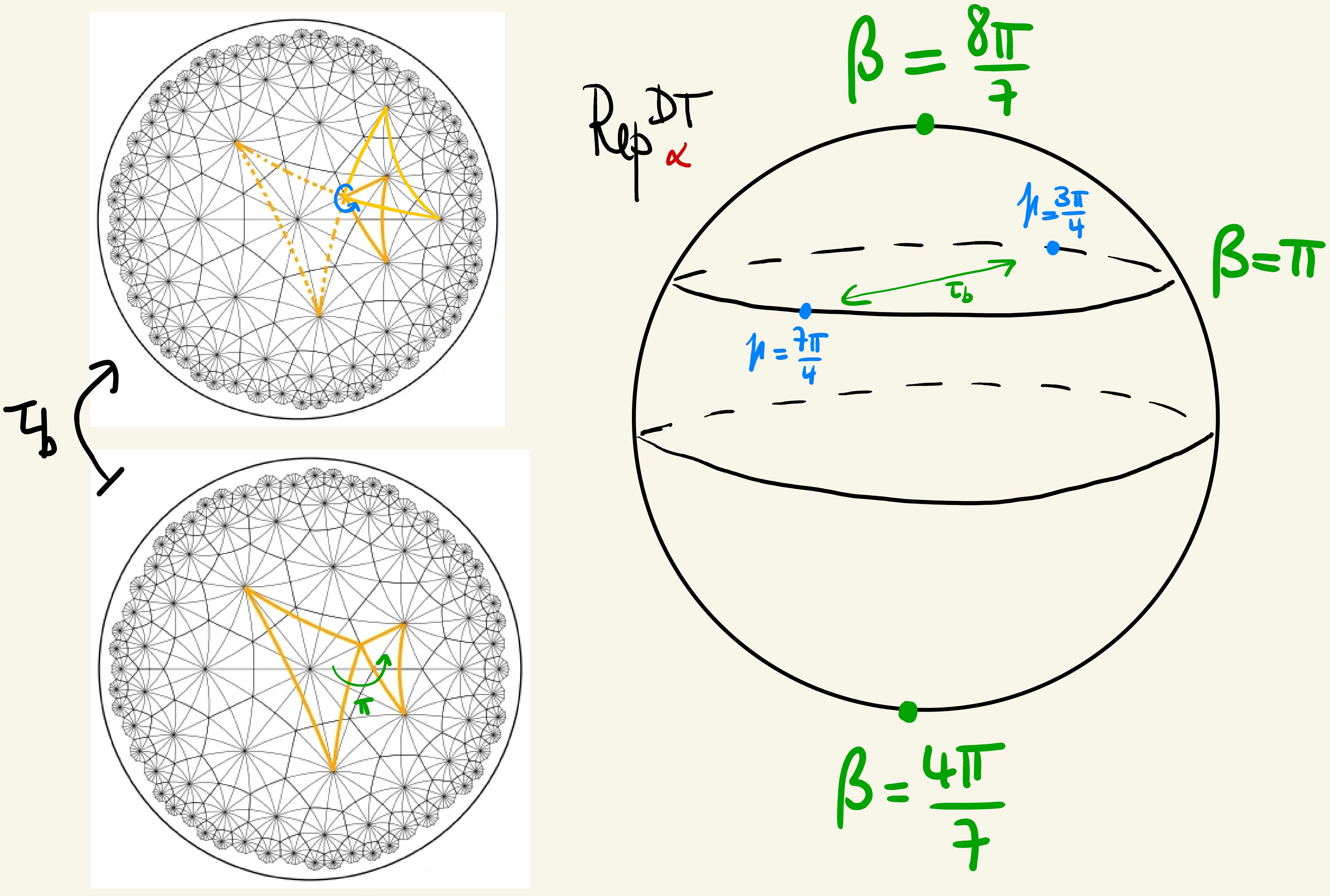
$\beta = \pi$

$\beta = \frac{4\pi}{7}$

n=4



$PMod(S) = \langle \tau_b, \tau_d \rangle$

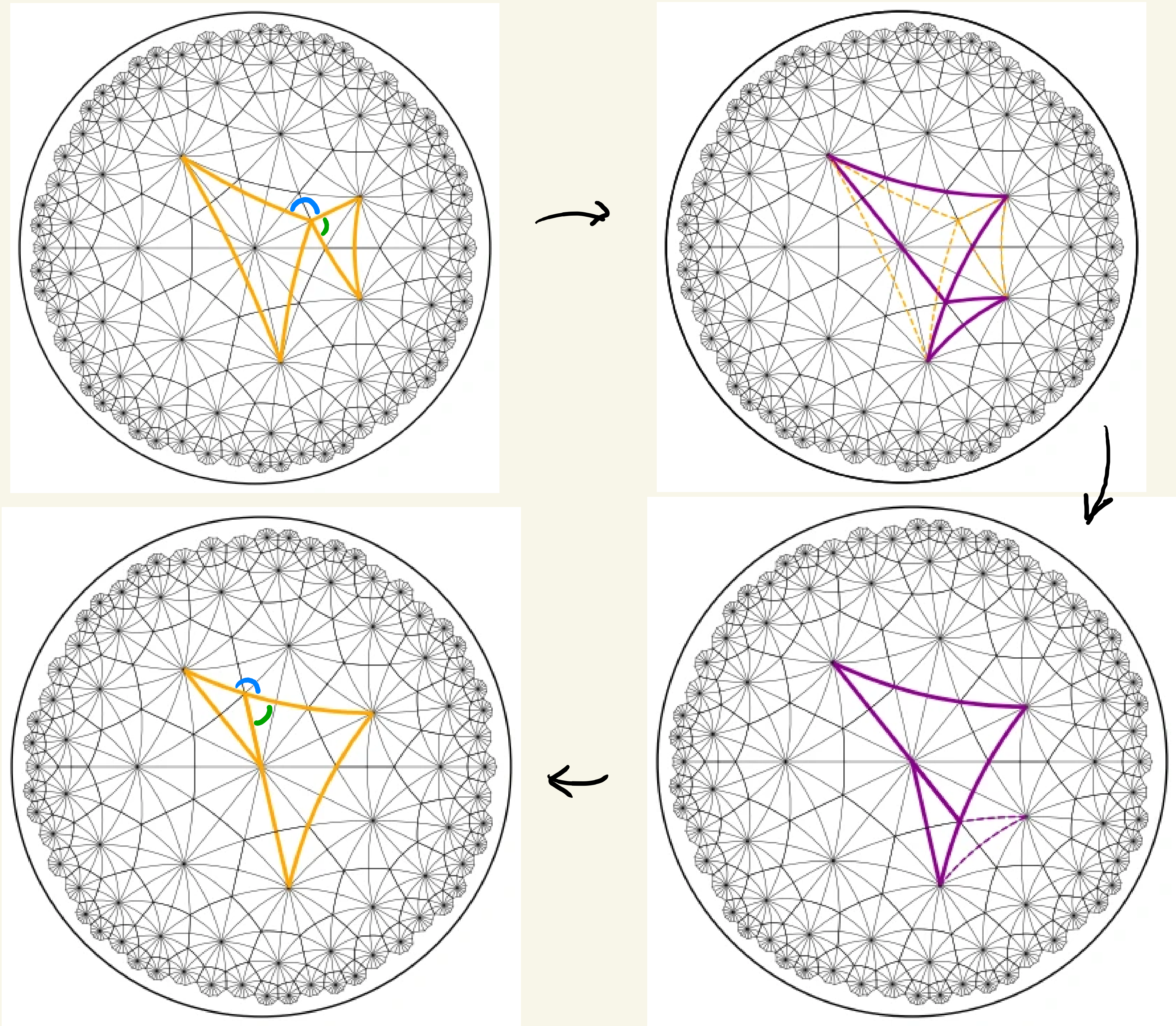


τ_b
π
$\mathrm{Rep}^{DT}_{\alpha}$
$\beta = \frac{8\pi}{7}$
$\mu = \frac{3\pi}{4}$
$\beta = \pi$
τ_b
$\mu = \frac{7\pi}{4}$
$\beta = \frac{4\pi}{7}$

$\beta = \pi$

$\gamma = \frac{3\pi}{4}$

τ_d



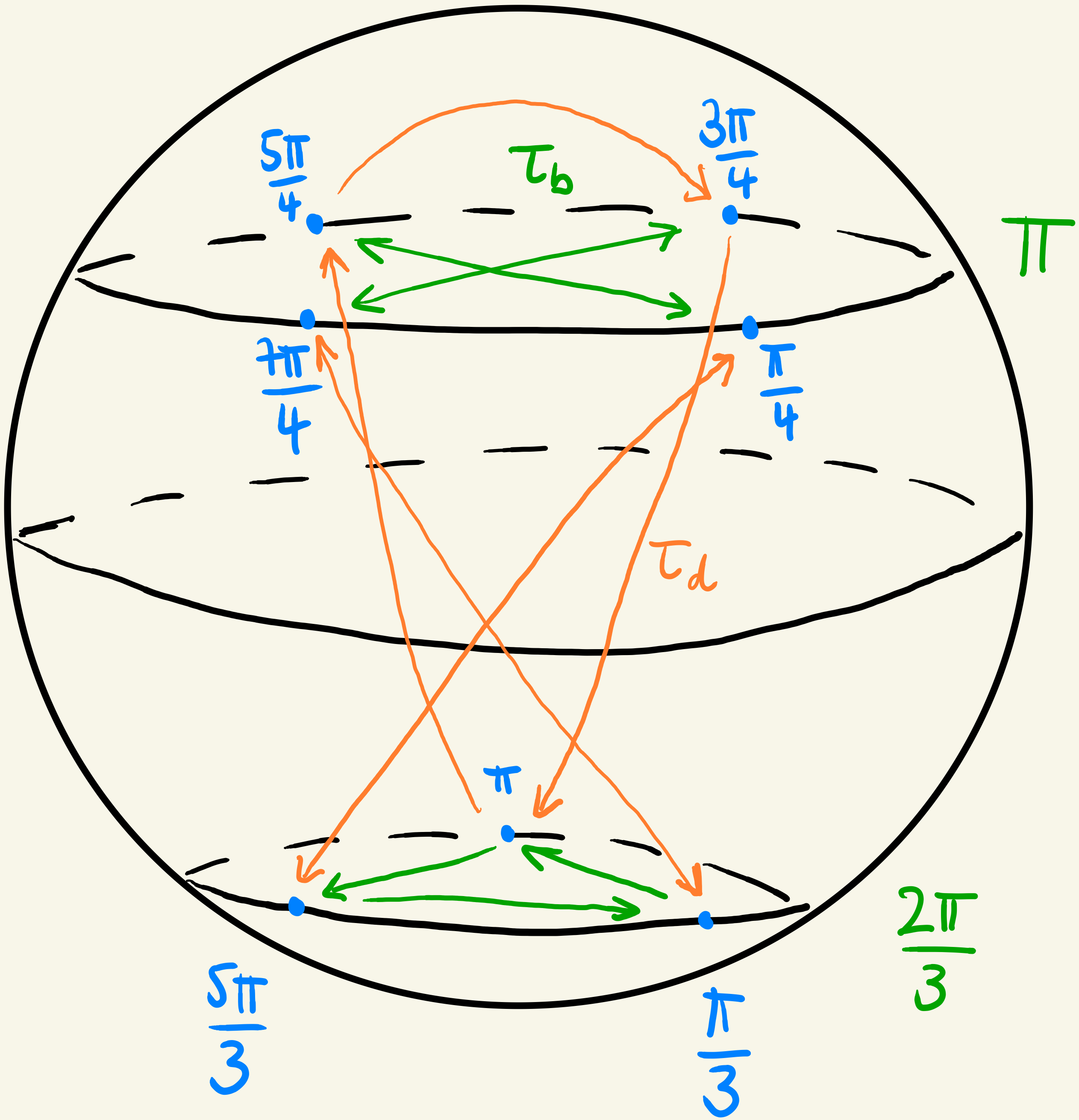
$\beta = \frac{2\pi}{3}$

$\gamma = \pi$

length = 7

"Klein orbit"

(Boalch 05')



Lisovyy-Tykhyy's classification	Orbit length	Angle vector α	Non-peripheral trace field
Sol. II	2	$\{\theta_1, \theta_1, \theta_2, \theta_2\}, \quad \theta_1 + \theta_2 > 3\pi$	
Sol. III	3	$\{\frac{4\pi}{3}, 2\theta - 2\pi, \theta, \theta\}, \ \theta > 5\pi/3$	
Sol. IV	4	$\{\pi, \theta, \theta, \theta\}, \ \theta > 5\pi/3$	
Sol. IV*	4	$\{\theta, \theta, \theta, 3\theta - 4\pi\}, \ \theta > 5\pi/3$	
Sol. 1	5	$\{\frac{22\pi}{15}, \frac{8\pi}{5}, \frac{8\pi}{5}, \frac{28\pi}{15}\}$	$\mathbb{Q}$
Sol. 4	6	$\{\frac{19\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{23\pi}{12}\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 6	6	$\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 7	6	$\{\frac{17\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}, \frac{29\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 8	7	$\{\frac{10\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\}$	$\mathbb{Q}$
Sol. 10	8	$\{\frac{17\pi}{12}, \frac{7\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 11	8	$\{\frac{13\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}, \frac{19\pi}{10}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 12	8	$\{\frac{3\pi}{2}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{19\pi}{10}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 13	9	$\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 14	9	$\{\frac{14\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 15	10	$\{\frac{8\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}, \frac{9\pi}{5}\}$	$\mathbb{Q}$
Sol. 18	10	$\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{23\pi}{15}, \frac{9\pi}{5}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 19	10	$\{\frac{7\pi}{5}, \frac{29\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 20	12	$\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 22	12	$\{\frac{19\pi}{15}, \frac{9\pi}{5}, \frac{9\pi}{5}, \frac{29\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 23	12	$\{\frac{37\pi}{30}, \frac{47\pi}{30}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 24	12	$\{\frac{49\pi}{30}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{59\pi}{30}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 25	12	$\{\frac{43\pi}{30}, \frac{49\pi}{30}, \frac{53\pi}{30}, \frac{59\pi}{30}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 26	15	$\{\frac{8\pi}{5}, \frac{26\pi}{15}, \frac{26\pi}{15}, \frac{26\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 27	15	$\{\frac{6\pi}{5}, \frac{28\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 30	16	$\{\frac{7\pi}{4}, \frac{7\pi}{4}, \frac{7\pi}{4}, \frac{7\pi}{4}\}$	$\mathbb{Q}$
Sol. 32	18	$\{\frac{37\pi}{21}, \frac{37\pi}{21}, \frac{37\pi}{21}, \frac{41\pi}{21}\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 33	18	$\{\frac{4\pi}{3}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 34	18	$\{\frac{25\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 37	20	$\{\frac{47\pi}{30}, \frac{53\pi}{30}, \frac{19\pi}{10}, \frac{19\pi}{10}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 38	20	$\{\frac{41\pi}{30}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{59\pi}{30}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 39	24	$\{\frac{3\pi}{2}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 40	30	$\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 41	30	$\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 43	40	$\{\frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 44	40	$\{\frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 45	72	$\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$	$\mathbb{Q}(\sqrt{5})$

TABLE 4. Lisovyy-Tykhyy's classification of finite orbits of DT representations.

Sol. II	2	$\{\theta_1, \theta_1, \theta_2, \theta_2\}, \quad \theta_1 + \theta_2 > 3\pi$
Sol. III	3	$\{\frac{4\pi}{3}, 2\theta - 2\pi, \theta, \theta\}, \ \theta > 5\pi/3$
Sol. IV	4	$\{\pi, \theta, \theta, \theta\}, \ \theta > 5\pi/3$
Sol. IV*	4	$\{\theta, \theta, \theta, 3\theta - 4\pi\}, \ \theta > 5\pi/3$
Sol. 1	5	$\{\frac{22\pi}{15}, \frac{8\pi}{5}, \frac{8\pi}{5}, \frac{28\pi}{15}\}$
Sol. 4	6	$\{\frac{19\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{23\pi}{12}\}$
Sol. 6	6	$\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}\}$
Sol. 7	6	$\{\frac{17\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}, \frac{29\pi}{15}\}$
Sol. 8	7	$\{\frac{10\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\}$

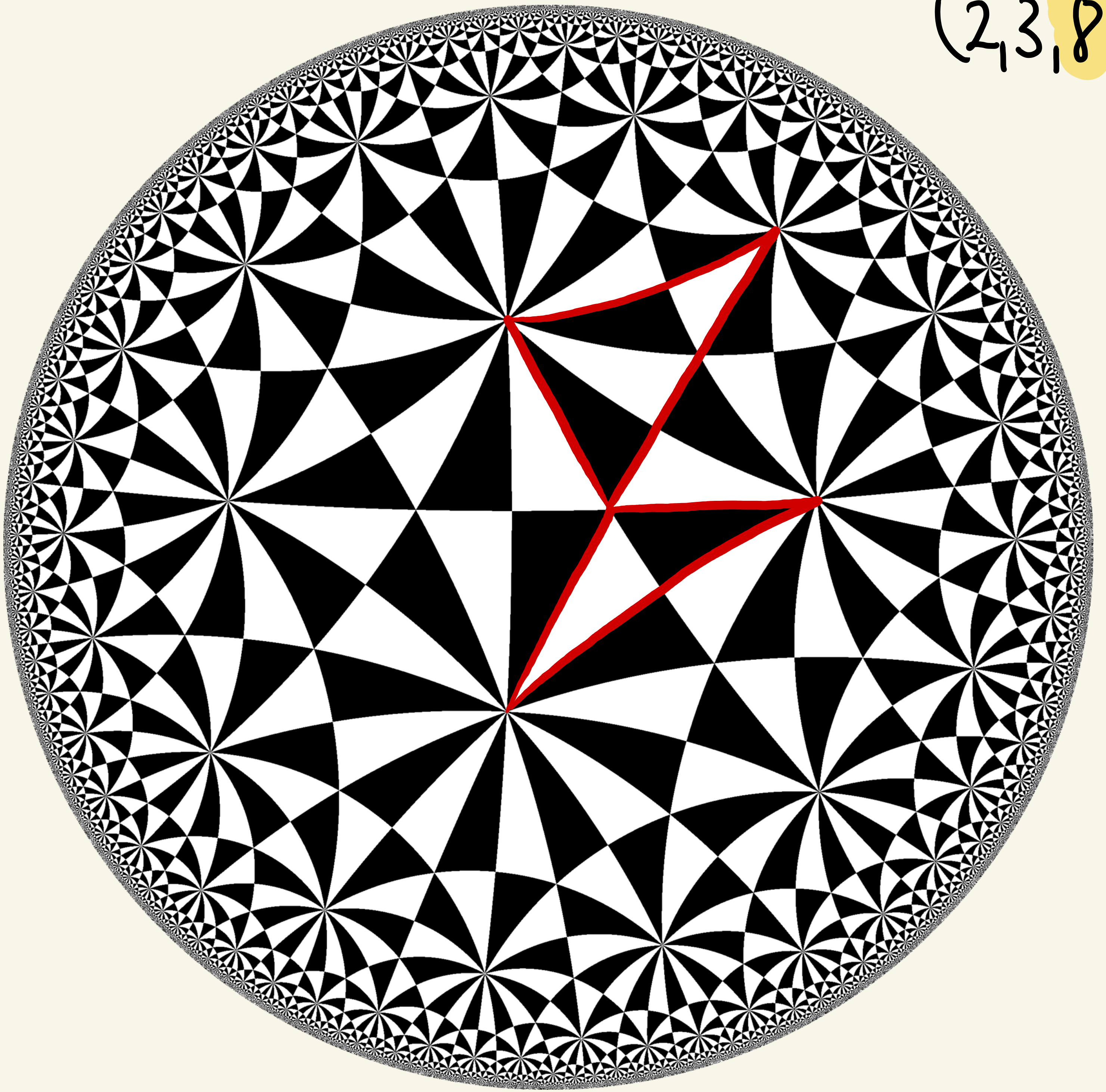
Sol. 34	18	$\{\frac{25\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}\}$
Sol. 37	20	$\{\frac{47\pi}{30}, \frac{53\pi}{30}, \frac{19\pi}{10}, \frac{19\pi}{10}\}$
Sol. 38	20	$\{\frac{41\pi}{30}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{59\pi}{30}\}$
Sol. 39	24	$\{\frac{3\pi}{2}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$
Sol. 40	30	$\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\}$
Sol. 41	30	$\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\}$
Sol. 43	40	$\{\frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}\}$
Sol. 44	40	$\{\frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}\}$
Sol. 45	72	$\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\}$

(2, 3, 8)

length = 4

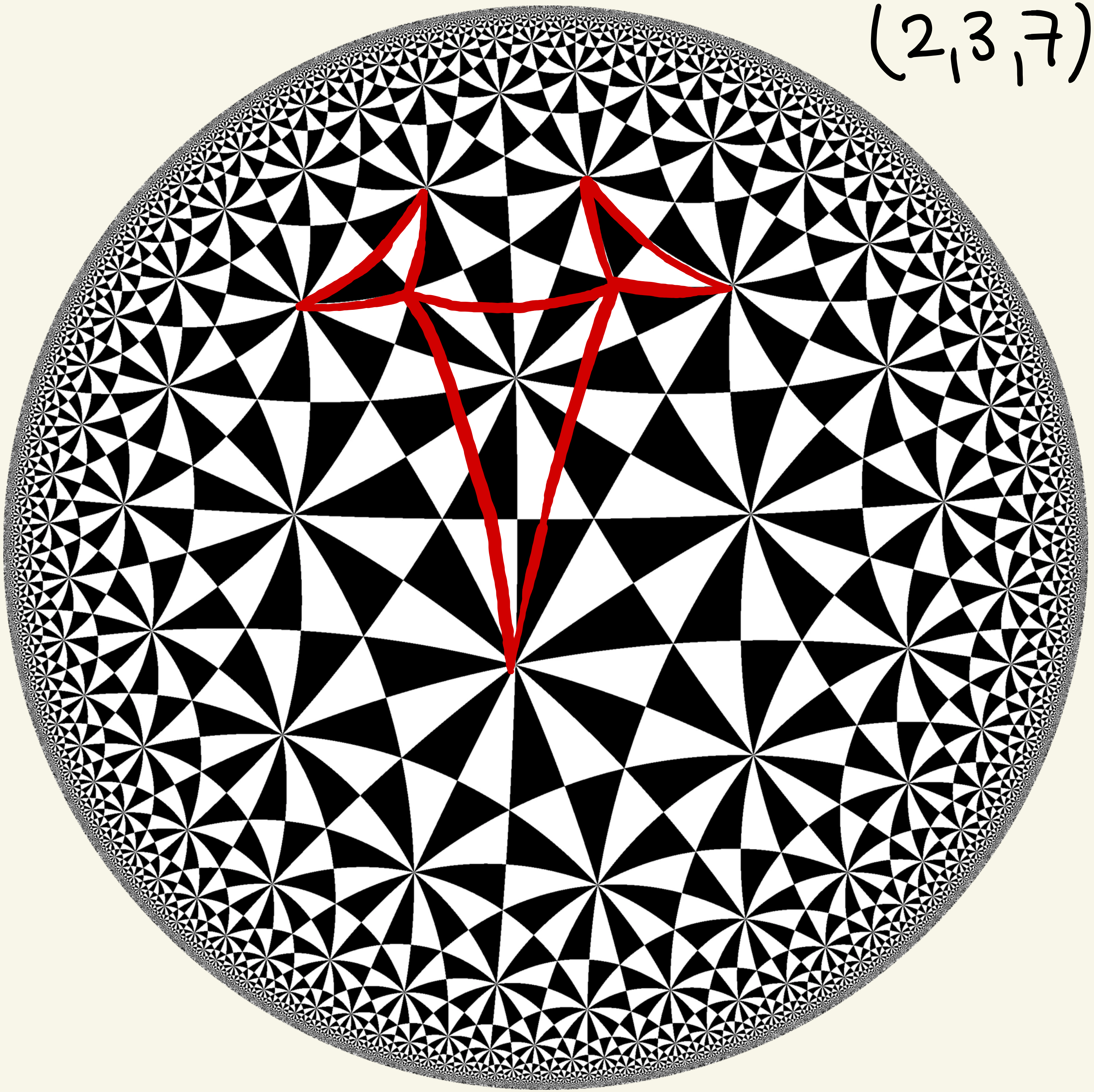
Type $\underline{IV}^*$

(Dubrovin 96')



$n=5$

length $=105$

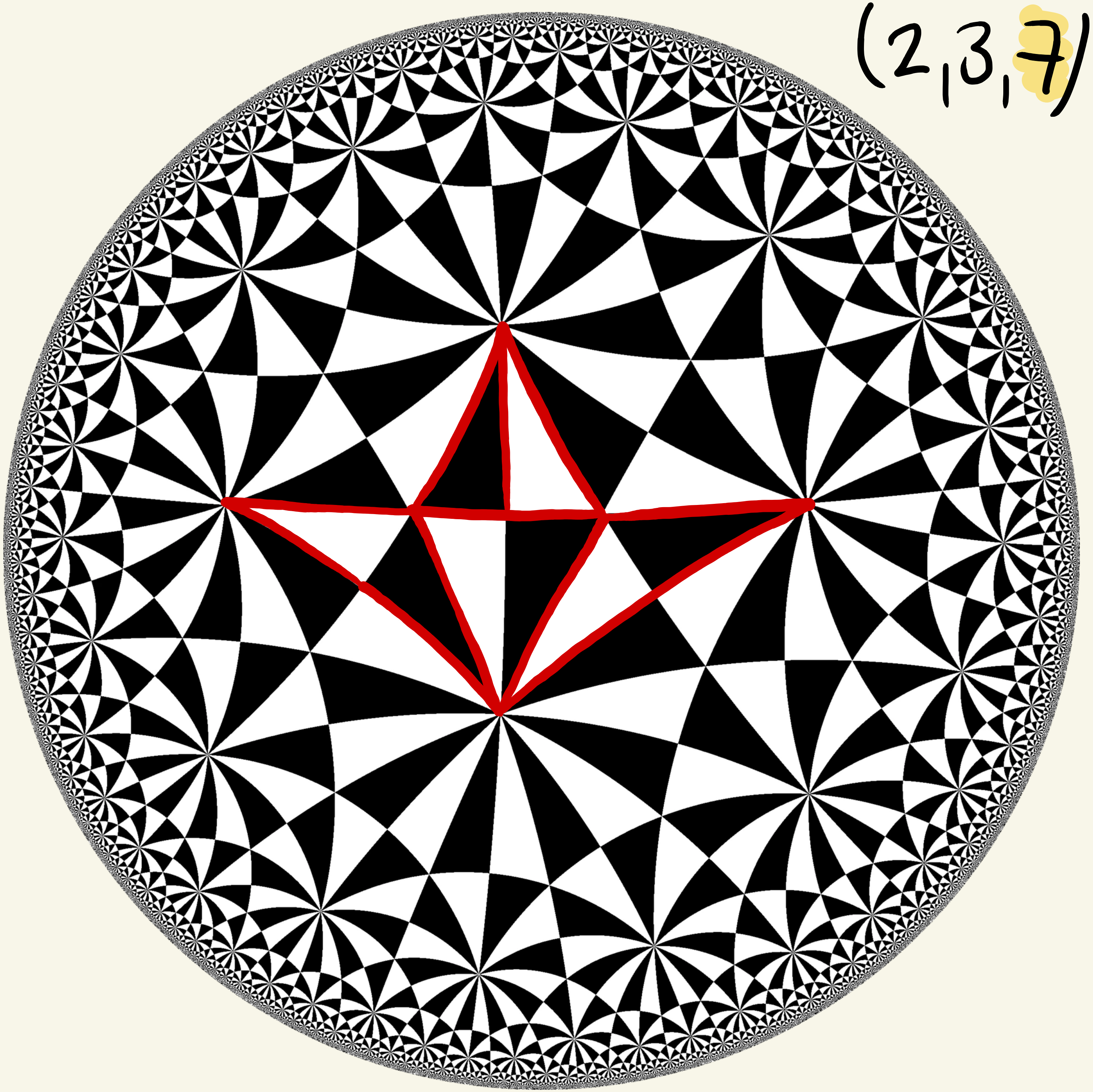


$(2,3,7)$

$(2,3,7)$

$n = 6$

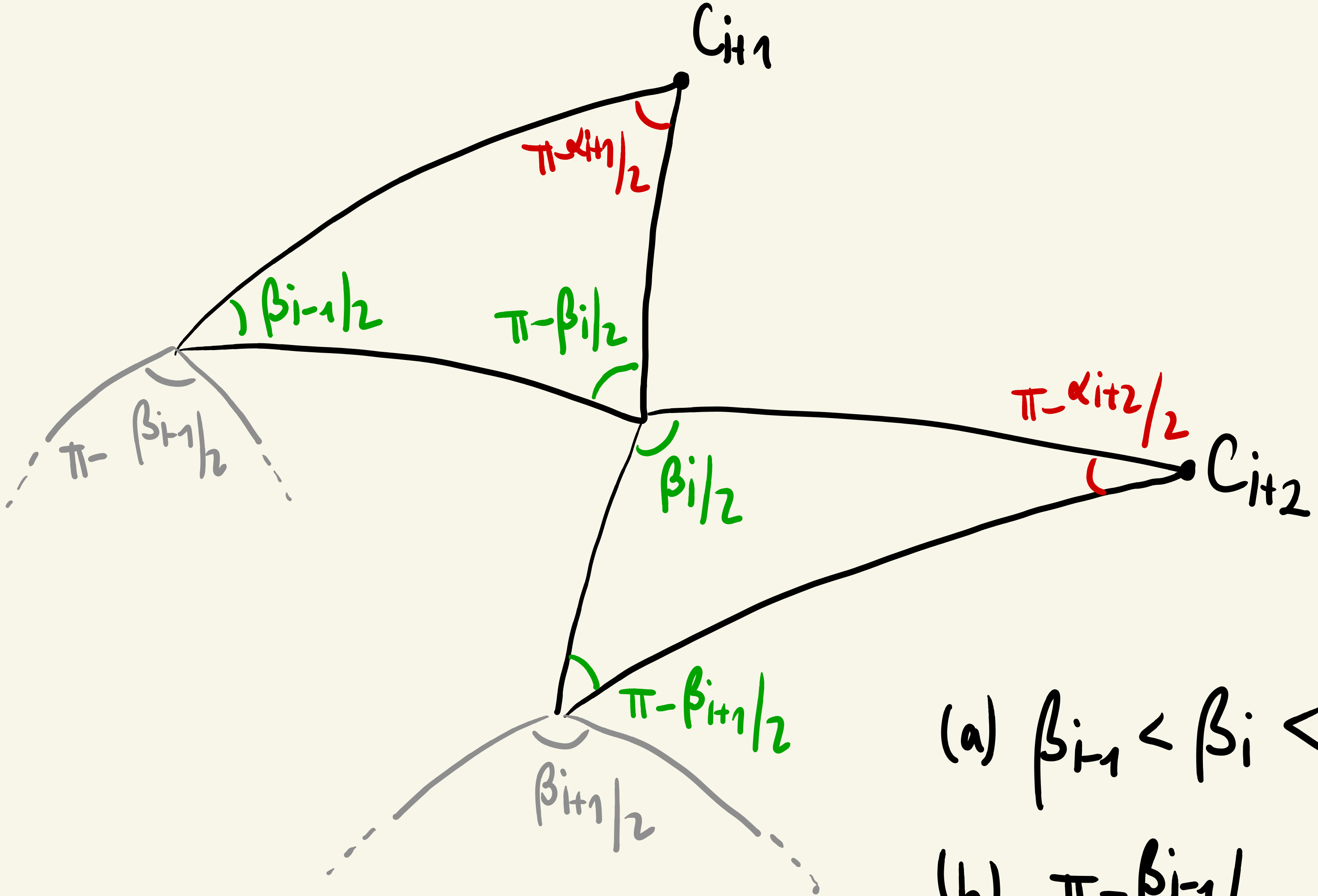
length $= 40$



Classifying finite orbits arising from triangle chains

(not necessarily discrete)

* 2 triangles ($n=4$) $\rightsquigarrow$ Boalch / Lisovyy–Tykhyy
* $n-2$ triangles ($n \geq 5$)
  - each consecutive pair of triangles belongs to B/LT's list



(a) $\beta_{i-1} < \beta_i < \beta_{i+1}$

(b) $\pi - \beta_{i-1}/2$, $\beta_{i-1}/2$ are both exterior angles

$\leadsto$ a chain has at most 4 triangles, i.e. $n \leq 6$

Thank you!