# YOUNG RESEARCHERS' WORKSHOP ON POSITIVITY IN LIE GROUPS

The main reference for the Young Researchers' Workshop on Positivity in Lie Groups is the paper from Guichard and Wienhard [GW18].
The following is an incomplete list of topics to be discussed. It additionally contains some references for each topic. We divided the topics we want to cover in four parts.

## 1. Higher Teichmüller spaces

This part is meant as a motivation and should give a bigger picture of how positivity arises in the context of representations of surface groups. You do not need to give all definitions, computational details or proofs. As a starting point, we recommend the main reference [GW18, Section 5] and a talk given by Anna Wienhard at MSRI in 2019 [Wie]. More detailed references are the surveys [Wie18] and [BIW14]. The following topics should be covered:

- Teichmüller space as a connected component of the character variety
- Surface group representations into semisimple Lie groups, generalized character variety
- Higher Teichmüller spaces (see [GW18, Section 5], [BIW14], [Wie18]); how does positivity appear in this context?
- Maximal representations into Lie groups of Hermitian type – give the definition and explain where positivity appears ([BIW14, Sections 5 and 7.1])
- Hitchin representations; existence of a positive flag curve (see [Lab06, Theorem 4.1] for the existence, [FG06, Theorem 1.15] for positivity)
- Make things explicit for Teichmüller space – What is the flag curve and what does positivity mean here? (Maybe talk to the people covering the positive order on the circle in Part 3)
- Positive representations ([GW18, Section 5] – Give the definition and state the conjectures. What is the role of positivity in the context of higher Teichmüller spaces? How does this lead to a common generalization of positivity as seen for maximal and Hitchin representations?
- Comparison between known (higher) Teichmüller spaces, e.g. for $\mathrm{PSL}(2,\mathbb{R})$, the space of Hitchin representations and of maximal representations agree with classical Teichmüller space; both Hitchin and maximal representations are positive; symplectic group as example for being real split and of Hermitian type – remark that Hitchin representations are maximal
- *optional:* geometric interpretation of some higher Teichmüller spaces: $\mathrm{PSL}(2,\mathbb{R})$: hyperbolic structures, $\mathrm{PSL}(3,\mathbb{R})$: convex real projective structures ([BIW14, Section 8.3]), the mapping class group of the surface acts properly on the Hitchin component ([Lab08]), see also [Kas18].

We know that some of the references (e.g. [FG06]) are very long and hard to read. We do not expect you to read all of it – just to give us a rough idea about the results and the concepts that are relevant to us.

## 2. Lie theory

The goal of this part is to lie the foundations to understand $\Theta$-positivity – the following list may be incomplete, so please scan [GW18, Section 4] for all concepts that are necessary there.
We want to mention the following references to learn about Lie theory: [Hum75], [Hum78] (good), [FH91] (good with many examples), [Hal03] (basic), [Kna02] (very detailed and more advanced), [Bor91] (advanced, old language), [Ser06]. Here is what we think should be covered:

- Definition of Lie group, Lie algebra, Killing form, the exponential map, representations of Lie groups and Lie algebras, adjoint representations (see [Hum75], . . . ), Cartan decomposition
- Semisimplicity, reductiveness
- Borel and parabolic subgroups, Levi subgroup, tori, (unipotent) radical
- What do parabolic subgroups and flags have to do with each other? (used in [GW18, Section 4.4])
- Representation of Lie algebras, (highest) weights, weight lattice, roots, root system, Weyl groups, Weyl chambers (see e.g. [Hum78])
- Dynkin diagrams, classification of semisimple Lie groups
- Symmetric spaces (of Hermitian type) in the context of Lie theory
- Real split Lie groups, Lie groups of Hermitian type

In particular, the examples are very important to us. If you cannot or do not want to cover everything in the general setting, you can explain the cases $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(4,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$ and maybe $\mathrm{SO}(p,q)$ for small $p, q$. They will appear in the next talks and will be crucial for us to understand what is going on. An option could be to work out the examples in exercise sessions. It would be nice to have at least one example where we see **all** the abstract definitions from above explicitly – from the definition of the Lie group, over the root system to the Dynkin diagram.

## 3. Notions of positivity

This part introduces us to different notions of positivity in various contexts. It should underline the parallels between the concepts, that later will be unified in the definition of $\Theta$-positivity. It follows [GW18, Sections 2 and 3] and the following concepts and questions could be discussed:

- Positive reals and the order on circle ([GW18, Section 2.1]):
  - What does positivity on the circle have to do with positivity of matrices?
  - Example $\mathrm{SL}(2,\mathbb{R})$
- Total positivity for matrices ([GW18, Section 2.2]):
  - decomposition theorem ([Whi52], [Loe55])
  - Example $\mathrm{SL}(3,\mathbb{R})$
  - parametrization of the subsemigroups $U^{>0}, O^{>0}$

  - optional: applications ([And87])
- Lusztig's total positivity in the context of real split Lie groups ([Lus94] – hard to read...)
  - What are $U, O$ for $\mathrm{Sp}(2n, \mathbb{R})$?
  - Some more references: [Lus08], [Lus98b], [Lus98a]
  - Double Bruhat cells and Total positivity ([FZ99])
- Positivity for Lie groups of Hermitian type ([GW18, Section 2.3]):
  - Example $\mathrm{Sp}(2n, \mathbb{R})$
  - Other examples of Lie groups of Hermitian types – what are the domains for $\mathrm{SU}(n, n)$, $\mathrm{SO}(2, n)$?
  - compare $\mathrm{Sp}(n, \mathbb{R})^{>0}$ and $\mathrm{Sp}(n, \mathbb{R})^{\succ 0}$, i.e. the two notions of positivity for $\mathrm{Sp}(n, \mathbb{R})$
- Positivity of triples in flag varieties ([GW18, Section 3.2]):
  - definition of flags and positivity of flag triples
  - some technicalities, e.g. transitive action of $\mathrm{SL}(n, \mathbb{R})$ on pairs of transverse flags
  - positive $n$-tuples, in particular quadruples ([FG07], [FG06])
  - show that positivity for triples in terms of Bonahon-Dreyer coordinates ([BD14], [BD17]) is equivalent to this notion
  - How is positivity of triples in a generalized flag variety $G/B$ defined? Example, e.g. $\mathrm{Sp}(4, \mathbb{R})$?
- Positivity and Maslov index ([GW18, Section 3.2]):
  - example $\mathrm{Sp}(2n, \mathbb{R})$
  - underline the analogy to ([GW18, Section 2.2])
  - some technicalities: transitive action on $\mathcal{L}$ and on pairs of transverse Lagrangians
  - definiton of Maslov index, generalized Maslov index ([Cle04], [COr01]), causal structure

As in the part about Lie groups, we encourage you to do explicit examples in small dimension in detail (possibly in exercise sessions).

## 4. Θ-positivity

Finally, we get to define Θ-positivity, following [GW18, Section 4]. You could cover the following topics/answer the following questions:

- short repetition of the structure of parabolic subgroups ([GW18, p.8]), definition and structure of weight spaces ([GW18, p.9])
- Why can we write $\beta$ uniquely as restriction of a root? Why is $u_\beta$ an irreducible representation of $L_\Theta$? Why is $u_\beta$ invariant under the action of $L_\Theta$? Give examples for e.g. $\mathrm{SL}(3, \mathbb{R})$, $\mathrm{Sp}(4, \mathbb{R})$, $\mathrm{SL}(4, \mathbb{R})$ (here, there can be more than one $\beta$, decomposable and indecomposable)
- explain [GW18, Example 4.1]
- give an example for every family of Lie groups that admit a Θ-positive structure as in Theorem 4.3 (except probably the exceptional ones) – e.g. for $\mathrm{SL}(3, \mathbb{R})$ show that there does not exists an $L_\Theta^\circ$-invariant sharp convex cone if $\Theta \neq \Delta$
- Definition of Θ-positive structure
- verify that $\mathrm{SL}(n, \mathbb{R})$ has a Δ-positive structure and that $\mathrm{Sp}(4, \mathbb{R})$ has a $\{\alpha_2\}$-positive structure using Example 4.1

- Theorem 4.3 including proof (or the idea)
  - for the proof see [Ben00](french)
  - define or recall *proximal* and *highest weight in* $2P$
  - explain the connection to Dynkin diagrams
  - show the Dynkin diagrams for the cases from the Theorem and explain when $\Theta$ does (not) satisfies the necessary condition
- connection to nonnegative subsemigroups (end of Sections 4.2 and Section 4.3)
  - explain Example 4.4 in more detail
  - for Section 4.3, underline the analogy to Section 2.2, with Weyl group replacing the symmetric group (which is the Weyl group for $\mathrm{SL}(n, \mathbb{R})$)
- $\Theta$-positive subsemigroup
  - explain $W(\Theta)$ for the different cases,
  - parametrization of the subsemigroup, underline analogy to Section 2.2
  - Theorem 4.5 – proof not necessarily, maybe explicitly for $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{SO}(3, q)$; the idea follows [BZ97]
- Why is it a subsemigroup?
- Explicit example: Subsemigroup for $\mathrm{Sp}(4, \mathbb{R})$ – what is it? Compare to the end of Section 2.3
- definition of $\Theta$-positive triples in flag variety ([GW18, Section 4.4] and geometric interpretation of $\Theta$-positivity (here, the reference does not include proofs); parabolic subgroups vs. flags
- State the Conjectures 4.10-4.12; maybe explain them for $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{Sp}(4, \mathbb{R})$ (cf. remark at the end of Section 4.4)
- Do the example $\mathrm{SO}(3, q)$ in detail (either explain it or as an exercise), see [GW18, Section 4.5]
- $\Theta$-positive representations as higher Teichmüller spaces ([GW18, Sections 5.3, 5.4]) – remember the motivation from Part 1.

More references can be found in the main reference [GW18].

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