Exo 3 partie 2

$$X(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Γ : cercle centré en $(2,0)$ de rayon 1

$\gamma(t) = (2+\cos(t), \sin(t))$ avec $t \in [0, 2\pi)$

$\gamma'(t) = (-\sin(t), \cos(t))$

$$X(\gamma(t)) = \left(\frac{-\sin(t)}{5+4\cos(t)}, \frac{2+\cos(t)}{5+4\cos(t)} \right)$$

$$\int_\gamma \langle X, d\gamma \rangle = \int_0^{2\pi} \langle X(\gamma(t)), \gamma'(t) \rangle \, dt$$

$$= \int_0^{2\pi} \frac{1+2\cos(t)}{5+4\cos(t)} \, dt$$

changement de variable : $u = \tan(t/2) \Leftrightarrow t = 2\arctan(u)$

$$dt = \frac{2}{1+u^2}$$

rappel : $\cos(t) = \frac{1-u^2}{1+u^2}$

$$\frac{1+2\cos(t)}{5+4\cos(t)} \, dt = \frac{1+2\frac{1-u^2}{1+u^2}}{5+4\frac{1-u^2}{1+u^2}} \, \frac{2}{1+u^2} \, du$$

$$= \frac{2(3-u^2)}{(9+u^2)(1+u^2)} \, du$$

décomposition en fractions élémentaires :

$$\frac{2(3-u^2)}{(9+u^2)(1+u^2)} \overset{!}{=} \frac{A}{9+u^2}+\frac{B}{1+u^2} = \frac{(A+9B)+(A+B)u^2}{(9+u^2)(1+u^2)}$$

$$\begin{cases} A+9B = 6 \\ A+B = -2 \end{cases} \Leftrightarrow \begin{cases} A = -3 \\ B = 1 \end{cases}$$

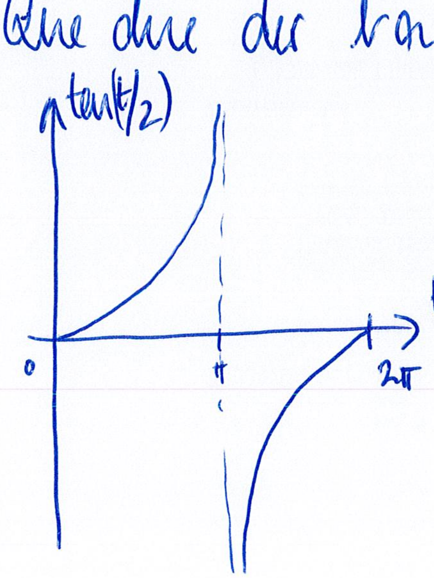
Donc $\displaystyle\int \frac{1+2\cos(t)}{5+4\cos(t)}\,dt = \int \frac{-3}{9+u^2}+\frac{1}{1+u^2}\,du$

$$= \int \frac{1}{1+u^2}\,du - \underbrace{\int \frac{3}{9+u^2}\,du}_{\substack{\text{on pose } u=3v \\ du = 3dv}}$$

$$= \arctan(u) - \int \frac{1}{1+v^2}\,dv$$

$$= \arctan(u) - \arctan(u/3)$$

Que dire des bornes?



- t varie entre 0 et 2π
- $u = \tan t/2$ varie entre $-\infty$ et $+\infty$

Autrement dit, quitte à retirer π, $\tan\left(\frac{t}{2}\right) : [0, 2\pi)\setminus\{\pi\} \to (-\infty, +\infty)$ est une bijection.

Ainsi $\displaystyle\int_0^{2\pi} \frac{1+2\cos(t)}{5+4\cos(t)}\, dt$

$$= \Big[\arctan(u) - \arctan(u/3)\Big]_{-\infty}^{+\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{2} - \left(-\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= 0.$$