*Workshop on*

# Character varieties under the lens of Hodge theory

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**Abstract.** Character varieties of surface groups are rich geometric objects lying at the intersection of topology, representation theory, and complex geometry. This workshop explores character varieties through the lens of Hodge theory, with a particular emphasis on complex variations of Hodge structures and their incarnation via non-abelian Hodge theory. Beginning with an introduction to polarized complex variations of Hodge structures, the workshop develops the correspondence between Hodge-theoretic data, Higgs bundles, and surface group representations. We will investigate how the $\mathbb{C}^*$-action on Higgs bundle moduli spaces detects representations arising from variations of Hodge structures, leading to geometric and dynamical applications to character varieties of closed and punctured surfaces. Topics include branched hyperbolic structures, compact components of relative character varieties, Hodge groups and their classification, and rigidity phenomena such as the Corlette–Simpson alternative. The workshop aims to make Hodge-theoretic techniques accessible while highlighting recent advances and open problems at the interface of Hodge theory and character varieties.

## Contents

# 1 Introductory talks

## 1.1 Introduction to complex variations of Hodge structures (Jean Douçot)

The goal of this first talk is to understand what is a *polarized complex variation of Hodge structures*. A possible approach could be to first recall what is the Hodge decomposition of the cohomology of a compact Kähler manifold, as a motivation. Then one could define complex variations of Hodge structures ($\mathbb{C}$-VHS), explaining the meaning of and reasons for Griffiths' transevrsality, as well as polarizations of $\mathbb{C}$-VHS. We'll focus on $\mathbb{C}$-VHS, but feel free to mention $\mathbb{Q}$-VHS and $\mathbb{R}$-VHS too, and how they are different from $\mathbb{C}$-VHS. Emphasize both points of views of Hodge decompositions and Hodge filtrations and explain why flatness matters. One should also define period maps and period domains. Ideally, one could illustrate these abstract definitions by working out one or two examples of $\mathbb{C}$-VHS explicitly. Do not restrict to the case of $\mathbb{C}$-VHS over Riemann surfaces (versus general Kähler manifolds), but briefly mention what changes/becomes simpler in that case. Make sure the talk is accessible to a large audience of geometers who are not necessarily experts in complex or algebraic geometry.

Robles' notes are a concise presentation of the topic with the same point of view as we wish to follow. Voisin's book, on the other hand, is a more complete and detailed reference.

**Suggested references:**


1. Colleen Robles, Hodge Theory: A representation theoretic perspective. `https://sites.math.duke.edu/~robles/22.06-Paris.pdf`

2. Claire Voisin, Hodge Theory and Complex Algebraic Geometry I (Chapters 2 and 3).


## 1.2 The moduli space of Higgs bundles over closed Riemann surfaces (Junming Zhang)

The goal of this talk is to construct the moduli space of Higgs bundles over closed Riemann surfaces. Define Higgs bundles and provide examples. Explain how to get a Higgs bundle from a $\mathbb{C}$-VHS. Introduce the relevant notions of stability and use them to define the moduli space of Higgs bundles. Two options of references include Guichard's and Wentworth's notes.

A key notion to present is the action of $\mathbb{C}^*$ on the moduli space of Higgs bundles and its interpretation as the gradient flow of the so-called energy functional. One should explain the equivalence between fixed points of the $\mathbb{C}^*$-action and the Higgs bundles coming from a $\mathbb{C}$-VHS. Another relevant observation to mention is that all fixed points have nilpotent Higgs fields, but having a nilpotent Higgs field doesn't imply being fixed by the $\mathbb{C}^*$-action. Some time could be spent explaining the Byalinicki-Birula Theorem (no need to give a detailed proof, but present it as a useful tool) and how it can be used to decompose the tangent space to the moduli space of Higgs bundle at a fixed point of the $\mathbb{C}^*$-action into "stable" and "unstable" loci.

If time permits, one could also define the Hitchin fibration and explain its properness.

**Suggested references:**


1. Olivier Guichard, An introduction to the differential geometry of flat bundles and of Higgs bundles, https://irma.math.unistra.fr/ guichard/assets/files/intro-bdle-ims.pdf

2. Richard Wentworth, Higgs bundles and local systems over Riemann surfaces, https://arxiv.org/pdf/1402.4203

3. A. Bialynicki-Birula, Some Theorems on Actions of Algebraic Groups (original paper).

4. James B. Carrell, Torus actions and cohomology. In Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action (description of the Byalinicki-Birula Theorem).

## 1.3 Closed surface groups representations and the case of $\mathrm{SL}_2\mathbb{C}$ (Farid Diaf)

The starting point of this talk is the non-abelian Hodge (NAH) correspondence between moduli spaces of Higgs bundles and character varieties of surface group representations. For simplicity, you should only consider *closed* surfaces throughout this talk. Make sure to define irreducible and reductive representations, and explain how to use these notions to define character varieties. State the NAH correspondence (you can also briefly mention the proof but don't spend too much time on it) for semisimple complex Lie groups first, as well as for their real forms in a second time. You may restrict to the case of $\mathrm{SL}_n\mathbb{C}$ and its real forms $\mathrm{SU}(p,q)$ if it makes the exposition simpler.

Using the NAH correspondence, we may say that a surface group representation *comes from a* $\mathbb{C}$*-VHS* if there exists a Riemann surface structure for which the corresponding Higgs bundle is fixed by the $\mathbb{C}^*$-action. The goal of the talk is to explain why surface group representations into $\mathrm{SL}_2\mathbb{C}$ that come from a $\mathbb{C}$-VHS in the above sense are either unitary representations (kind of the trivial case) or are holonomies of *branched hyperbolic surfaces*.

Explain that it's an open question to determine which closed surface group representations into $\mathrm{PSL}_2\mathbb{R}$ are holonomies of branched hyperbolic structures. It may be wroth to recall some results on the topology of $\mathrm{PSL}_2\mathbb{R}$-character varieties for closed surfaces, and the notion of Euler number. You can then show examples of representations that are holonomies of branched hyperbolic structures (like all Fuchsian representations and others in the branched case), and examples of representations that aren't. Make sure to state Goldman's Conjecture on this question.

**Suggested references:**

1. Brice Loustau, Minimal surfaces and quasi-fuchsian structures https://brice.loustau.eu/ressources/Asheville.pdf (especially Section 3).
2. William M. Goldman, Mapping class group dynamics on surface group representations (especially Section 3.7).

## 1.4 Parabolic Higgs bundles and compact components of relative character varieties (Sophia Bugarija)

This talk is somewhat similar to the previous one, except that it focuses on surfaces with punctures. The relevant notion to introduce is that of *parabolic Higgs bundles*. Define the moduli space of parabolic Higgs bundles, as well as the notion of *relative* character varieties, and state the NAH correspondence in this context. Briefly mention that we still have a $\mathbb{C}^*$-action and say that we can similarly define what it means for a surface group representation to come from a $\mathbb{C}$-VHS.

The goal of the talk is present several examples of compact components of relative character varieties for genus-0 surface group representations. We are particularly interested in the examples of Deroin–Tholozan, and the later generalizations by Tholozan–Toulisse and Feng–Zhang. For simplicity, you may only consider the cases where the targets groups are $\mathrm{SU}(p,q)$. Explain that all these compact components are made of *universal* $\mathbb{C}$*-VHS*, i.e. representations that come from $\mathbb{C}$-VHS for all choices of complex structures on the underlying surface. For the examples of Deroin–Tholozan, you may also explain the hyperbolic geometric description of those representations as holonomies of maximally branched hyperbolic spheres with cone singularities.

Ideally, on could also present how Wu recovers the above examples of compact components by considering *minimal energy local systems*.

**Suggested references:**

1. Claudio Meneses, Notes on Parabolic Higgs bundles. https://www.math.uni-kiel.de/geometrie/de/claudio-meneses/parabolic-higgs-bundles
2. B. Deroin and N. Tholozan, Supra-maximal representations from fundamental groups of punctured surfaces to $\mathrm{PSL}_2\mathbb{R}$.

3. N. Tholozan and J. Toulisse, Compact connected components in relative character varieties of punctured spheres.
4. Y. Feng and J. Zhang, Compact relative $\mathrm{SO}_0(2,q)$-character varieties of punctured spheres.
5. Charlie Wu, Minimal Energy Local Systems on Curves.

### 1.5 Hodge groups: different points of view and classification (Nazim Khelifa)

The goal of this talk is to understand the following three equivalent notions for connected semi-simple Lie groups $G$.

1. $G$ is a *Hodge group*, meaning that $G$ is the real Zariski closure of the image of a surface group representations that come from a $\mathbb{C}$-VHS.
2. $G$ has a compact Cartan subgroup.
3. $G$ admits open subsets of elliptic elements, meaning that the subset of elliptic elements in $G$—those that belong to a compact subgroup of $G$ (sometimes called compact elements)—has non-empty interior.

Examples of such groups include of Lie groups of Hermitian types (which you can define), but not only; for instance, $\mathrm{SO}(p,2q)$ is a Hodge group for all integers $p$, $q$. After explaining why the three notions are equivalent, you can give the complete list of Hodge groups, and give as many details about the classification as time permits. You may also explain the relation with Kähler groups described in Py's lecture notes.

**Suggested references:**

1. Pierre Py, Lectures on Kähler groups (especially Chapter 10 and references therein).
2. M. I. Kabenyuk, Compact elements and Cartan subgroups of connected Lie groups.
3. Mitsuo Sugiura, Conjugate classes of Cartan subalgebras in real semisimple Lie algebras.

### 1.6 The Corlette–Simpson alternative (Vasily Rogov)

Based on Corlette–Simpson, *On the classification of rank two representations of quasiprojective fundamental groups*, the goal of this talk is to present the Corlette–Simpson alternative. It concerns representations from the fundamental group of a quasi-projective variety (like the moduli space of curves for instance) with quasi-unipotent monodromy at infinity into $\mathrm{SL}_2\mathbb{C}$. We are interested in a modern formulation of the alternative that states that those representations are either

- non-rigid and come from a map to an orbicurve,
- or they are rigid. In that case, they have integral monodromy and come from a $\mathbb{C}$-VHS.

Make sure to state the alternative as precisely as possible. Focus will be given on the proof of integrality for rigid representations into $\mathrm{SL}_2\mathbb{C}$.

Also mention that by the work of Loray–Pereira–Touzet, the hypothesis of quasi-unipotent monodromy at infinity may be dropped. By their Theorem A, all the representations with non quasi-unipotent monodromy at infinity come from a map to an orbicurve.

**Suggested references:**

1. K. Corlette and C. Simpson, On the classification of rank-two representations of quasiprojective fundamental groups.
2. Frank Loray, Jorge Vitório Pereira, and Frédéric Touzet, Representations of quasi-projective groups, flat connections and transversely projective foliations.

### 1.7 Holomorphic isomonodromic leaves of the joint moduli space of Higgs bundles (Enya Hsiao)

The goal of this talk is to understand the isomodromic foliation of the joint moduli space of Higgs bundles recently introduced by Collier–Toulisse–Wentworth. Start by introducing all relevant moduli spaces and their relevant geometric structures. Emphasis should be put on understanding the results of Section 1.1: namely, the relation between the (non-)holomorphicity of the isomonodromic foliation, the (non-)degeneracy of the Hermitian/symplectic form on the joint moduli of Higgs bundles, and the nature of the corresponding surface group representations (Θ-positive/Anosov representations versus totally elliptic representations). For instance, Theorem E and Corollary 1.8 relate to the fourth talk on compact components of relative character varieties.

**Suggested references:**

1. B. Collier, J. Toulisse, and R. Wentworth, Higgs bundle, isomonodromic leaves and minimal surfaces.

## 2 Research talks

### 2.1 Mirror symmetry of subregular downward flows (Ana Peón-Nieto)

In this talk, I will focus on Hodge bundles of subregular type. These fixed points can be classified into very stable and wobbly depending on the dynamics of the C* action. After motivating the interest of these objects, I will turn to the role of very stable fixed points of subregular type in brane mirror symmetry. The downward flows of these objects yield a BAA brane, a conjectural mirror of which I will propose.

### 2.2 Modelling hyper-Kähler structures on moduli of parabolic Higgs bundles over the Riemann sphere (Claudio Meneses )

The non-abelian Hodge correspondence is a deep analytic result lying behind the existence of natural hyper-Kähler structures on moduli of parabolic Higgs bundles on compact Riemann surfaces. These structures arise in an infinite series of families, in part as a consequence of their dependence on choices of stability parameters. By the very nature of their construction, their characterisation beyond existence is a nontrivial problem, and although the wall-crossing phenomena associated with such dependence is well understood as a problem in birational geometry, the analogous differential-geometric problem for the hyper-Kähler structure is still outstanding.

In this talk I will present an overview of recent work on the construction of geometric models that could yield a more explicit dependence of the hyper-Kähler structure on stability parameters in the case of genus 0. In particular I will discuss results obtained in collaboration with Lynn Heller and Sebastian Heller, on how a suitable renormalised limit of the Hitchin metrics converge to the hyperpolygon-space metrics as the stability parameters are scaled down to 0.

### 2.3 Poisson geometry of irregular parabolic Higgs bundles (Marina Logares)

We study the Poisson geometry of moduli spaces of irregular parabolic Higgs bundles over a smooth complex projective curve, allowing higher order poles along a non-reduced effective divisor, as well as the associated integrable system.

### 2.4 From cyclic Higgs bundles to Anosov representations (Qiongling Li)

Anosov representations generalize Fuchsian representations of surface groups, retaining key properties such as discreteness and faithfulness. Constructing new families of Anosov representations remains difficult. Via nonabelian Hodge theory, surface group representations can be studied through Higgs bundles. In this talk, I will present several families of cyclic Higgs bundles whose associated representations are Anosov, yielding new closed families of Anosov representations. This is based on work in progress with Samuel Bronstein and Colin Davalo.