A note on character varieties

Arnaud Maret

amaret@mathi.uni-heidelberg.de

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Contents

Foreword

The material presented is this note[1] is classical. The notes are an extended version of a minicourse I gave in the Spring 2022 at the KIT Karlsruhe and at the University of Heidelberg. The aim is to introduce the notions of *representation* and *character varieties*, taking into account various approaches found in the literature. We cover both the analytic and algebraic perspectives and insist on the symplectic geometry aspects of character varieties at the end of the notes. Most of it is inspired from [Sik12], [Mon16, §2], [Lab13], and [BGPGW07].

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Chapter 1

Representation varieties

A representation variety is an analytic, sometimes algebraic, object associated to a finitely generated group Γ and a Lie group G. It consists of the space of group homomorphisms from Γ to G. We start by recalling some generalities about Lie groups, including algebraic groups, and finitely generated groups. Most of the results later in this note require to restrict the groups Γ and G to finer classes. The relevant notions are presented in the next section.

1.1 Setting: Lie groups and finitely generated groups

1.1.1 Lie groups

A *Lie group* G is a real smooth manifold with a group structure for which the operations of multiplication and inverse are smooth maps. Lie groups always admit an analytic atlas, unique up to analytic diffeomorphism, such that multiplication and inverse are analytic maps[1]. Lie groups are not necessarily connected. We denote by G° the identity component of G. The *centralizer* of a subset $S \subset G$ is denoted $Z(S) := \{g \in G : gsg^{-1} = s, \forall s \in S\}$. It is a closed subgroup of G and hence a Lie subgroup of G. The standard examples of Lie groups are $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, and all their closed subgroups, called *linear Lie groups*, which include $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SU}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}(n, \mathbb{R})$.

The Lie algebra of a Lie group G is denoted $\mathfrak{g}$. Most of the time, we will think of $\mathfrak{g}$ as the tangent space to G at the identity. In various places we will make use of the Lie theoretic exponential map $\exp\colon \mathfrak{g} \to G$, which, in the case that G is a linear Lie group, is the matrix exponential map. The adjoint representation of G on $\mathfrak{g}$ is denoted by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ and is defined by

$$\mathrm{Ad}(g)(\xi) := \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1}, \quad g \in G,\, \xi \in \mathfrak{g}.$$

A Lie algebra $\mathfrak{g}$ is

- *simple* if it is not abelian and if its only proper ideal is the zero ideal. Since ideals of $\mathfrak{g}$ are in one-to-one correspondence with sub-representations of its adjoint representation, $\mathfrak{g}$ is simple

[1] This is a consequence of the Campbell-Hausdorff formula, see e.g. [Ser06, Chap. IV, §7-8]

if and only if its adjoint representation is irreducible and $\mathfrak{g}$ is not a one-dimensional abelian Lie algebra.

- *semisimple* if it has no nonzero abelian ideals. Equivalently, a Lie algebra is semisimple if it is a direct sum of simple Lie algebras [Bou98, Chap. I, §6.2, Cor. 1]. By Cartan's criterion, $\mathfrak{g}$ is semisimple if and only if its *Killing form*

$$\begin{aligned} K\colon \mathfrak{g}\times\mathfrak{g} &\to \mathbb{R} \\ (\xi_1,\xi_2) &\mapsto \operatorname{Tr}(\operatorname{ad}(\xi_1)\operatorname{ad}(\xi_2)) \end{aligned}$$

  is nondegenerate [Bou98, Chap. I, §6.1, Thm. 1].

- *reductive* if it is the direct sum of an abelian and a semisimple Lie algebra. Equivalently, $\mathfrak{g}$ is reductive if and only if its adjoint representation is completely reducible[2], which is further equivalent to $\mathfrak{g}$ admitting a faithful, completely reducible, finite-dimensional representation [Bou98, Chap. I, §6.4, Prop. 5].

We call a connected Lie group *simple, semisimple* or *reductive* if its Lie algebra is simple, semisimple or reductive, respectively. Simple Lie groups are semisimple and semisimple Lie groups are reductive. The groups $\mathrm{SL}(n,\mathbb{R})$ for $n\geqslant 2$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SU}(p,q)$ for $p+q\geqslant 2$ are simple. The group $\mathrm{SO}(n,\mathbb{R})^\circ$ is simple for $n\geqslant 3, n\neq 4$ and semisimple for $n=4$. In contrast, the group $\mathrm{GL}(n,\mathbb{R})^\circ$ is not semisimple for any $n\geqslant 1$ (its Killing form is degenerate). It is however reductive, because its Lie algebra is the direct sum of the simple Lie algebra of traceless matrices and the abelian Lie algebra of diagonal matrices. It is worth observing that a connected linear Lie group $G\subset \mathrm{GL}(n,\mathbb{R})$ is reductive if and only if the trace form

$$\begin{aligned} \operatorname{Tr}\colon \mathfrak{g}\times\mathfrak{g} &\to \mathbb{R} \\ (\xi_1,\xi_2) &\mapsto \operatorname{Tr}(\xi_1\xi_2) \end{aligned}$$

is nondegenerate. This can be seen as a consequence of the classification of semisimple Lie algebras and [Bou98, Chap. I, §6.4, Prop. 5]. The previous statement also holds for connected linear Lie groups $G\subset\mathrm{GL}(n,\mathbb{C})$. If the (in this case, complex-valued) trace form is nondegegenrate, then so is its real part $\Re(\operatorname{Tr})\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ which gives a nondegenerate, symmetric, Ad-invariant, real-valued bilinear form.

A Lie group is called a *complex Lie group* if it has the structure of a complex manifold and the group operations are holomorphic. Standard examples of complex Lie groups include $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$.

1.1.2 Quadrable Lie groups

An important class of Lie groups for the purpose of this work are those that admit a nondegenerate, symmetric and Ad-invariant pairing on their Lie algebra. Such Lie groups carry different names throughout the literature, see [Ova16] for an overview. We opt for the name quadrable.

[2] Recall that a *completely reducible* representation is a representation that decomposes as a direct sum of irreducible representations. Such representations are sometimes called *semisimple*.

Definition 1.1.1 (Quadrable Lie groups)**.** A Lie group G is called *quadrable* if there exists a bilinear form (also called pairing)

$$B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

which is nondegenerate, symmetric and Ad-invariant.

Quadrable Lie groups are common among the standard Lie groups. For example, all semisimple Lie groups, and more generally all reductive Lie groups, are quadrable. Indeed, a nondegenerate, symmetric and Ad-invariant bilinear form on a reductive Lie algebra can be taken to be the Killing form on the semisimple part and any nondegenerate, symmetric bilinear form on the abelian part. Alternatively, one may consider the trace form associated to a faithful, finite-dimensional representation[3] of $\mathfrak{g}$. We point out that not all quadrable Lie groups are reductive, see [Gol84, Footnote p. 204].

Example 1.1.2. For instance, $G = \mathrm{SL}(2,\mathbb{R})$ is quadrable. We usually chose to work with the pairing given by the trace form: $\mathrm{Tr}\colon \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R} \to \mathbb{R}$, $(\xi_1, \xi_2) \mapsto \mathrm{Tr}(\xi_1\xi_2)$. The trace of a matrix is invariant under conjugation, so the trace form is Ad-invariant. In the basis

$$\mathfrak{sl}_2\mathbb{R} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

the trace form is given by the pairing $2x_1x_2 + y_1y_2 + z_1z_2$. It is clearly symmetric and nondegenerate. Actually, in this case, the pairing $\mathrm{Tr}\colon \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$ is also positive-definite.

Example 1.1.3. The Heisenberg group H is an example of a non-quadrable Lie group. Recall that H is defined to be the group of strictly upper triangular 3×3 real matrices:

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{h}$ of H is generated by the three matrices

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that Z commutes with any element of H. Further

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(X) = X - Z, \quad \mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(Y) = Y, \tag{1.1.1}$$

[3] The *trace form of a representation* $\rho\colon \mathfrak{g} \to \mathrm{GL}(n,\mathbb{R})$ is the symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $(\xi_1, \xi_2) \mapsto \mathrm{Tr}(\rho(\xi_1)\rho(\xi_2))$. For instance, the Killing form is the trace form of the adjoint representation.

and

$$\mathrm{Ad}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}(X)=X,\quad \mathrm{Ad}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}(Y)=Y+Z. \tag{1.1.2}$$

So, because of (1.1.1), any symmetric and Ad-invariant bilinear form $B\colon \mathfrak{h}\times\mathfrak{h}\to\mathbb{R}$, must satisfy

$$B(X,Z)=B(X-Z,Z) \text{ and } B(X,Y)=B(X-Z,Y)$$

which implies $B(Z,Z)=0$ and $B(Y,Z)=0$. Moreover, because of (1.1.2), it must also satisfy

$$B(X,Y)=B(X,Y+Z)$$

and thus $B(X,Z)=0$. This shows that B is degenerate.

1.1.3 Algebraic groups

A group G is called an *algebraic group* if it is an algebraic variety[4] and if the operations are regular maps. The Zariski closure of any subgroup of G is an algebraic subgroup [Mil17, Lem. 1.40] and any algebraic subgroup of G is Zariski closed [Mil17, Prop. 1.41]. For instance, the centralizer $Z(S)$ of a subset $S\subset G$ is Zariski closed and hence an algebraic subgroup. All algebraic groups over the fields of real or complex numbers, respectively called *real or complex algebraic groups*, are also Lie groups, see [Mil13, III, §2] and references therein. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. The group $\mathrm{GL}(n,\mathbb{K})$, and all its Zariski closed subgroups, such as $\mathrm{SL}(n,\mathbb{K})$, $\mathrm{Sp}(2n,\mathbb{K})$ or $\mathrm{SO}(n,\mathbb{K})$, are algebraic groups. They are called *linear algebraic groups*. Algebraic groups, however, are not necessarily linear (for instance, elliptic curves are non-linear algebraic groups). The group $\mathrm{SU}(p,q)$ is a real algebraic group, but is not a complex algebraic variety, see e.g. [SKKT00, Exercise 1.1.2].

Any algebraic group contains a unique maximal normal connected solvable subgroup called the *radical*, see [Mil17, Chap. 6, §h]. A *reductive algebraic group* is a connected algebraic group whose radical over $\mathbb{C}$ is an algebraic torus, i.e. isomorphic to $(\mathbb{C}^*)^n$ for some $n\geqslant 0$. A reductive algebraic group over the fields of real or complex numbers is a reductive Lie group in the previous sense, hence quadrable [Mil13, II, §4].

Connected linear algebraic groups $G\subset \mathrm{GL}(n,\mathbb{C})$ are reductive if and only if the trace form $\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$, $(\xi_1,\xi_2)\mapsto \mathrm{Tr}(\xi_1\xi_2)$ is nondegenerate. In particular, $\mathrm{SL}(n,\mathbb{C})$ for $n\geqslant 2$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$ for $n\geqslant 3$ are reductive algebraic groups.

1.1.4 Finitely generated groups

The second ingredient of a representation variety is a finitely generated group Γ. Finitely generated groups are always equipped with the discrete topology. Our guiding example of finitely generated groups are surface groups.

[4]In the context of this work, an *algebraic variety* is understood to be the zero locus of a set of polynomial equations over $\mathbb{R}$ or $\mathbb{C}$ (in other words, algebraic varieties are always affine). We make no assumption about irreducibility and, in particular, we don't distinguish algebraic varieties and algebraic sets. Morphisms of algebraic varieties are restrictions of polynomial maps and are called *regular maps*.

Definition 1.1.4 (Surface group)**.** Let $g \geqslant 0$ and $n \geqslant 0$ be two integers. A group is called a *surface group* if it can presented as

$$\pi_{g,n} := \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \, : \, \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{n} c_j = 1 \right\rangle, \tag{1.1.3}$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator of a_i and b_i. If $n = 0$, then it is called a *closed surface group.*

The closed surface groups $\pi_{g,0}$ are pairwise non-isomorphic (because their cohomology with real coefficients differs in degree 1), non-free for $g \geqslant 1$ and non-abelian for $g \geqslant 2$. If $n \geqslant 1$, then the surface group $\pi_{g,n}$ is isomorphic to the free group on $2g + n - 1$ generators. The name "surface group" is explained by the following lemma.

Lemma 1.1.5. *Let $\Sigma_{g,n}$ denote a connected orientable topological surface of genus $g \geqslant 0$, with $n \geqslant 0$ punctures. The fundamental group of $\Sigma_{g,n}$ is isomorphic to $\pi_{g,n}$.*

Proof. The proof for the case $n = 0$ is explained in [Lab13, Thm. 2.3.15]. Its generalization to punctured surfaces can be understood in two steps. First, observe that a sphere with $n \geqslant 1$ punctures is homotopy equivalent to the wedge of $n - 1$ circles. Hence, its fundamental group is the free group on $n - 1$ generators. Similarly, a surface of genus g with one puncture is homotopy equivalent to the wedge of $2g$ circles. Thus, its fundamental group is the free group on $2g$ generators. Now, note that $\Sigma_{g,n}$ is the union of two sub-surfaces $\Sigma_{g,1}$ and $\Sigma_{0,n+1}$. The conclusion now follows from Van Kampen's Theorem. □

The generators c_i in (1.1.3) will play a central role later in Section 4.2 in the context of relative representation varieties. They should be thought of as homotopy classes of based loops enclosing the ith puncture of $\Sigma_{g,n}$.

1.2 Definition

Definition 1.2.1 (Representation variety)**.** The *representation variety* associated to a finitely generated group Γ and a Lie group G is the set of group homomorphisms from Γ to G and is denoted by

$$\mathrm{Hom}(\Gamma, G).$$

The elements $\phi \in \mathrm{Hom}(\Gamma, G)$ are called *representations.*

The topology on the representation variety $\mathrm{Hom}(\Gamma, G)$ is defined to be the subspace topology induced by the compact-open topology on the space G^Γ of all (necessarily continuous) functions $\Gamma \to G$.

Let $(\gamma_1, \ldots, \gamma_n)$ be a set of generators of Γ. We introduce the subspace

$$X(\Gamma, G) := \left\{ \left(\phi(\gamma_1), \ldots, \phi(\gamma_n)\right) : \phi \in \mathrm{Hom}(\Gamma, G) \right\} \subset G^n.$$

Lemma 1.2.2. *Let G be a Lie group equipped with an analytic atlas. The set $X(\Gamma, G)$ is an analytic subvariety*[5] *of G^n and is homeomorphic to* $\mathrm{Hom}(\Gamma, G)$. *In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural structure of analytic variety and the structure does not depend on the choice of generators of* Γ.

Proof. Let $R = \{r_i\}$ denote a (maybe infinite) set of relations for the generators $\gamma_1, \dots, \gamma_n$. Each relation r_i defines an analytic map $r_i \colon G^n \to G$ because multiplication and inverse are assumed to be analytic operations on G. The map r_i is called a *word map.* The set $X(\Gamma, G)$ is the analytic subset of G^n cut out by the relations $r_i(g_1, \dots, g_n) = 1$ for every i.

Since a group homomorphism $\phi \colon \Gamma \to G$ is determined by the images of a set of generators of Γ, the map

$$\begin{aligned} \Pi \colon \mathrm{Hom}(\Gamma, G) &\to X(\Gamma, G) \\ \phi &\mapsto \big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big) \end{aligned}$$

is a bijection. We prove that Π is a homeomorphism. Recall that all the sets

$$V(K, U) := \{f \colon \Gamma \to G : K \subset \Gamma \text{ finite}, U \subset G \text{ open}, f(K) \subset U\}$$

form a sub-basis for the compact-open topology on $\mathrm{Hom}(\Gamma, G)$. To see that Π is a continuous map, observe that, for a collection of open sets $U_1, \dots, U_n \subset G$,

$$\Pi^{-1}\left(X(\Gamma, G) \cap U_1 \times \ldots \times U_n\right) = \mathrm{Hom}(\Gamma, G) \cap \bigcap_{i=1}^{n} V(\{\gamma_i\}, U_i).$$

To prove that the inverse map Π^{-1} is also continuous, note that any element $k \in \Gamma$, seen as a word in the generators $\gamma_1, \dots, \gamma_n$, determines an analytic function $k \colon G^n \to G$. Now, given a finite set $K \subset \Gamma$ and an open set $U \subset G$, we have

$$\Pi\left(\mathrm{Hom}(\Gamma, G) \cap V(K, U)\right) = X(\Gamma, G) \cap \bigcap_{k \in K} k^{-1}(U).$$

We conclude that both Π and its inverse are continuous. Hence, Π is a homeomorphism.

If $(\gamma'_1, \dots, \gamma'_{n'})$ is another set of generators of Γ and $X'(\Gamma, G)$ is the associated space, then the map from $X(\Gamma, G)$ to $X'(\Gamma, G)$ defined as the composition

$$X(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) \to X'(\Gamma, G)$$

is an isomorphism of analytic varieties. Indeed, the map sends $\big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big)$ to $\big(\phi(\gamma'_1), \dots, \phi(\gamma'_{n'})\big)$. Now, since γ'_i is a word in the generators $\gamma_1, \dots, \gamma_n$, it follows that $\phi(\gamma'_i)$ is a word in $\phi(\gamma_1), \dots, \phi(\gamma_n)$. This shows that the map is analytic because word maps are analytic by assumption on G. $\square$

Lemma 1.2.3. *Assume that G has the structure of a real or complex algebraic group, then $X(\Gamma, G)$ is an algebraic subset of G^n. In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural structure of real or complex algebraic variety and the structure does not depend on the choice of generators of* Γ.

[5] An *analytic variety* is understood to be the zero locus of a set of analytic functions over $\mathbb{R}$ or $\mathbb{C}$.

Proof. The argument is analogous to the proof of Lemma 1.2.2. The key observation is that the relations $R = \{r_i\}$ give regular maps $r_i \colon G^n \to G$ by assumption on G. $\square$

Remark 1.2.4 (Finitely generated versus finitely presented)*.* Since we assumed Γ to be finitely generated, and not finitely presented, the set of equations that define $X(\Gamma, G)$ might be infinite. However, Hilbert's basis theorem implies that any algebraic variety over a field can be described as the zero locus of finitely many polynomial equations, see e.g. [SKKT00, §2.2].

Remark 1.2.5 (Standard topology versus Zariski topology)*.* If G is a real or complex algebraic group, then it is also a Lie group, as mentioned earlier. This means that the representation variety $\mathrm{Hom}(\Gamma, G)$ has both the structure of an analytic variety and of an algebraic variety. The underlying topology of the analytic structure is called the *standard topology* and that of the algebraic structure the *Zariski topology*. The standard topology on an algebraic variety is always Hausdorff. The Zariski topology is coarser than the standard topology. Indeed, Zariski open sets are open in the standard topology because polynomials are continuous functions. A nonempty Zarsiki open set is also dense in both the standard and the Zariski topology.

Example 1.2.6 (Surface groups)**.** Representations $\pi_{g,n} \to G$ typically arise as *holonomies* (or *monodromies*) of (G, X)-structures on $\Sigma_{g,n}$, see [Gol21] for further details. Not all the representations $\pi_{g,n} \to G$ are holonomies of (G, X)-structures. However, if $n = 0$, then the set of holonomies is an open subset of $\mathrm{Hom}(\pi_{g,0}, G)$ [Gol21, Cor. 7.2.2]. For instance, if $G = \mathrm{PSL}(2, \mathbb{R})$, then the holonomies of hyperbolic structures on the closed surface $\Sigma_{g,0}$, $g \geqslant 2$, are precisely the discrete and faithful representations in $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2, \mathbb{R}))$. They form two connected components of the representation variety.

In the vocabulary of category theory, we can say that representation variety is a bifunctor from the product of the category of finitely generated groups and the category of Lie/algebraic groups to the category of analytic/algebraic varieties. This is a consequence of Lemmata 1.2.2 and 1.2.3, and of the following.

Lemma 1.2.7. *Let Γ be a finitely generated group and G be a Lie/algebraic group.*

1. *If $\tau \colon \Gamma_1 \to \Gamma_2$ is a morphism of finitely generated groups, then the induced map $\tau^* \colon \mathrm{Hom}(\Gamma_2, G) \to \mathrm{Hom}(\Gamma_1, G)$ is an analytic/regular map.*

2. *If $r \colon G_1 \to G_2$ is a morphism of Lie groups or of algebraic groups, then the induced map $r_* \colon \mathrm{Hom}(\Gamma, G_1) \to \mathrm{Hom}(\Gamma, G_2)$ is an analytic map or a regular map, respectively.*

Proof. The second assertion is immediate. To prove the first statement, note that if $(\gamma_1^1, \ldots, \gamma_n^1)$ is a set of generators for Γ_1 and $(\gamma_1^2, \ldots, \gamma_m^2)$ is a set of generators for Γ_2, then $(\tau^*\phi)(\gamma_i^1) = \phi(\tau(\gamma_i^1))$ is a word in $\phi(\gamma_1^2), \ldots, \phi(\gamma_m^2)$. Word maps are analytic, respectively regular, and thus so is τ^*. $\square$

1.3 Symmetries

The representation variety $\mathrm{Hom}(\Gamma, G)$ has two natural symmetries given by the right action of the group $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ by pre-composition and the left action of $\mathrm{Aut}(G)$ by post-composition:

$$\mathrm{Aut}(G) \circlearrowright \mathrm{Hom}(\Gamma, G) \circlearrowleft \mathrm{Aut}(\Gamma).$$

An immediate consequence of Lemma 1.2.7 is

Lemma 1.3.1. *The actions of the groups* $\mathrm{Aut}(\Gamma)$ *and* $\mathrm{Aut}(G)$ *on* $\mathrm{Hom}(\Gamma, G)$ *preserve its analytic/algebraic structure.*

There is a normal subgroup of $\mathrm{Aut}(G)$ that is of particular interest: namely, the subgroup of inner automorphisms of G, denoted $\mathrm{Inn}(G)$. Recall that an *inner automorphism* of G is an automorphism given by conjugation by a fixed element of G. In particular, $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ denotes the centre of G (which is a closed and normal subgroup of G). The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is relevant in many concrete cases. For instance, the holonomy representations mentioned in Example 1.2.6 are really defined up to conjugation by an element of G and so it makes sense to see them as elements of the quotient

$$\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G). \tag{1.3.1}$$

The quotient (1.3.1) is the prototype of the notion of character variety introduced below.

The action of $\mathrm{Aut}(\Gamma)$ on the representation variety descends to an action of $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ on the quotient (1.3.1). The group $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is denoted $\mathrm{Out}(\Gamma)$ and is called the group of *outer automorphisms* of Γ.

Example 1.3.2 (Surface groups)**.** The group of outer automorphisms of the surface group $\pi_{g,n}$ has a particular significance. It contains the (pure) mapping class group of the surface $\Sigma_{g,n}$ as a subgroup. This is known as the Dehn–Nielsen Theorem. We develop this observation further in Section 6.2.

1.4 Zariski tangent spaces

In this section, we would like to determine the Zariski tangent spaces to representation varieties. We start by recalling the classical notion of Zariski tangent spaces for analytic varieties in $\mathbb{R}^n$.

Definition 1.4.1 (Zariski tangent spaces)**.** Let $X \subset \mathbb{R}^n$ is an analytic variety defined as the zero locus of some analytic functions $f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$. The *Zariski tangent space* at $x \in X$ is the kernel of the $m \times n$ Jacobi matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j}. \tag{1.4.1}$$

Equivalently, the Zariski tangent space at x consists of all tangent vectors $x'(0)$ tangent to a smooth path $x(t)$ inside $\mathbb{R}^n$ with $x(0) = x$ and that satisfies the relations $f_i = 0$ *up to first order* by which we mean that $f_i(x(0)) = 0$ and $\frac{d}{dt}\big|_{t=0} f_i(x(t)) = 0$.

To specialize to the case of representation varieties, we need a notion of Zariski tangent spaces for analytic varieties in the infinite product G^Γ. We follow the approach of [Kar92] and refer the reader to that paper for more details. The relevant notion here is that of real valued ringed space.

Definition 1.4.2 (Real valued ringed space)**.** A *real valued ringed space* is a topological space with a sheaf of real valued continuous functions.

Examples of real valued ringed spaces include smooth manifolds together with the sheaf of smooth functions, analytic varieties together with the sheaf of analytic functions or algebraic varieties together with the sheaf of rational maps. There is a notion of *Zariski tangent space* for real valued ringed spaces that generalizes the notion of tangent spaces for manifolds and that of Zariski tangent spaces for analytic and algebraic varieties.

On the space G^Γ, one can define a notion of smooth functions. A function $F\colon G^\Gamma \to \mathbb{R}$ is called *locally smooth* if it is locally a smooth function of a finite number of coordinates. The space G^Γ, together with the sheaf of locally smooth real-valued functions on G^Γ, is a real valued ringed space. In the case of G^Γ, the Zariski tangent space at any point can be identified with $\mathfrak{g}^\Gamma$ via left translation.

The representation variety $\mathrm{Hom}(\Gamma, G)$ is the subspace of the space G^Γ cut out by the equations

$$\phi(xy)\phi(y)^{-1}\phi(x)^{-1} = 1, \quad \forall x, y \in \Gamma.$$

As such, it has an induced ringed space structure. Previously, in the context of Lemma 1.2.2, we explained that $\mathrm{Hom}(\Gamma, G)$ inherits its structure from the embedding inside G^n that depends on a choice of generators for Γ. In contrast, the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$ does not require to fix a set of generators for Γ. The disadvantage is that G^Γ, unlike G^n, is an infinite product.

Lemma 1.4.3 ([Kar92])**.** *Fix a set of n generators of Γ and let F_n be the free group on n generators. The following diagram is a commutative diagram of real valued ringed spaces:*

$$\begin{tikzcd} & G^n \arrow[dr] & \\ \mathrm{Hom}(\Gamma, G) \arrow[ur] \arrow[dr] & & G^{F_n} \\ & G^\Gamma \arrow[ur] & \end{tikzcd}$$

In particular, the structures induced by G^n and G^Γ on $\mathrm{Hom}(\Gamma, G)$ coincide.

We refer the reader to [Kar92] for a proof of Lemma 1.4.3.

Working with the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$, we can determine the Zariski tangent space to the representation variety without referring to a presentation of Γ. Let $F_{x,y}\colon G^\Gamma \to G$ be defined by $F_{x,y}(f) := f(xy)f(y)^{-1}f(x)^{-1}$. The Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at ϕ is the intersection of the kernels of the linear forms $D_\phi F_{x,y}\colon \mathfrak{g}^\Gamma \to \mathfrak{g}$ for all $x, y \in \Gamma$ (each tangent space to G is naturally identified to $\mathfrak{g}$ via left translation).

Lemma 1.4.4. *It holds that*

$$D_\phi F_{x,y}(v) = v(xy) - v(x) - \mathrm{Ad}(\phi(x))v(y)$$

for $v \in \mathfrak{g}^\Gamma$ and $\phi \in \mathrm{Hom}(\Gamma, G)$.

Proof. By definition, we have that

$$\begin{aligned}
D_\phi F_{x,y}(v) &= \frac{d}{dt}\bigg|_{t=0} F_{x,y}(\exp(tv)\phi) \\
&= \frac{d}{dt}\bigg|_{t=0} \exp(tv(xy))\phi(xy)\phi(y)^{-1}\exp(-tv(y))\phi(x)^{-1}\exp(-tv(x)) \\
&= v(xy) - v(x) - \mathrm{Ad}(\phi(x))v(y).
\end{aligned}$$

Here $\exp\colon \mathfrak{g} \to G$ denotes the Lie theoretic exponential map. ☐

We conclude

Corollary 1.4.5 ([Gol84], [Kar92]). *The Zariski tangent space to* $\mathrm{Hom}(\Gamma, G)$ *at* ϕ *is*

$$T_\phi \mathrm{Hom}(\Gamma, G) = \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}.$$

Corollary 1.4.5 can be reformulated in terms of group cohomology[6]. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ equips $\mathfrak{g}$ with the structure of a Γ-module by

$$\Gamma \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}).$$

The resulting Γ-module is denoted by $\mathfrak{g}_\phi$. The set of 1-cochains in the bar complex that computes the cohomology of Γ with coefficients in $\mathfrak{g}_\phi$ is $\mathfrak{g}^\Gamma$, see Appendix B.2 for more details on the bar complex. The space of 1-cocycles is

$$Z^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}$$

and thus identifies with the Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at ϕ. The space of 1-coboundaries, defined by

$$B^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \mathrm{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\},$$

also plays a role in this context. They can be identified with the Zarisiki tangent space to the $\mathrm{Inn}(G)$-orbit of $\phi \in \mathrm{Hom}(\Gamma, G)$ at ϕ (recall from Section 1.3 that $\mathrm{Inn}(G)$ acts on the representation variety by post-composition). We denote this orbit by

$$\mathcal{O}_\phi \subset \mathrm{Hom}(\Gamma, G).$$

Proposition 1.4.6 ([Gol84], [Kar92]). *The Zariski tangent space to* $\mathcal{O}_\phi$ *at* ϕ *is*

$$T_\phi \mathcal{O}_\phi = \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \mathrm{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\}.$$

Proof. The orbit $\mathcal{O}_\phi$ is a smooth manifold isomorphic to the quotient of G by the stabilizer of ϕ for the conjugation action. The stabilizer of ϕ is the centralizer $Z(\phi) := Z(\phi(\Gamma))$ of $\phi(\Gamma)$ inside G, which is a closed subgroup of G. In particular, the Zariski tangent space to $\mathcal{O}_\phi$ at ϕ coincides with the usual notion of tangent space.

[6] We provide an introduction to group (co)homology, containing all the relevant notions for this work, in Appendix B.

A smooth deformation of ϕ inside $\mathcal{O}_\phi$ is of the form $\phi_t = g(t)\phi g(t)^{-1}$, where $g(t)$ is a smooth 1-parameter family inside G with $g(0) = 1$. The tangent vector to ϕ_t at $t = 0$ is the coboundary $v(x) = \xi - \mathrm{Ad}(\phi(x))\xi$ where $\xi \in \mathfrak{g}$ is the tangent vector to $g(t)$ at $t = 0$. Conversely, for any $\xi \in \mathfrak{g}$, the coboundary $v(x) = \xi - \mathrm{Ad}(\phi(x))\xi$ is tangent to $\exp(t\xi)\phi\exp(-t\xi)$ at $t = 0$. □

Observe that $B^1(\Gamma, \mathfrak{g}_\phi)$ can be identified with the quotient $\mathfrak{g}/\mathfrak{z}(\phi)$, where $\mathfrak{z}(\phi)$ is the Lie algebra of $Z(\phi)$. In particular, it holds that

$$\dim B^1(\Gamma, \mathfrak{g}_\phi) = \dim \mathcal{O}_\phi = \dim G - \dim Z(\phi). \tag{1.4.2}$$

We mention that the quotient

$$H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$$

is known as the first cohomology group of the group Γ with coefficients in the Γ-module $\mathfrak{g}_\phi$ introduced in Definition B.2.

Example 1.4.7 (Surface groups)**.** In the special case of a closed surface group, one can obtain the conclusion of Corollary 1.4.5 from the embedding $\mathrm{Hom}(\pi_{g,0}, G) \subset G^{2g}$. Let $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$ and let $A_i := \phi(a_i)$ and $B_i := \phi(b_i)$, where a_i and b_i are the generators of $\pi_{g,0}$ in the presentation (1.1.3). The Zariski tangent space to $\mathrm{Hom}(\pi_{g,0}, G)$ at ϕ is isomorphic to the kernel of the differential of the map

$$\begin{aligned} F\colon G^{2g} &\to G \\ (X_1, \dots, X_g, Y_1, \dots, Y_g) &\mapsto \prod_{i=1}^{g} [X_i, Y_i] \end{aligned}$$

at $(A_1, \dots, A_g, B_1, \dots, B_g)$. A simple computation shows that the kernel of $D_{(A_i,B_i)}F$ corresponds to the subset of $\mathfrak{g}^{2g}$ that consists of all those $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ such that

$$\begin{aligned} &\big(\alpha_1 + \mathrm{Ad}(A_1)\beta_1\big) - \mathrm{Ad}\left([A_1, B_1]\right)\big(\beta_1 + \mathrm{Ad}(B_1)\alpha_1\big) \\ &+ \mathrm{Ad}\left([A_1, B_1]\right)\big(\alpha_2 + \mathrm{Ad}(A_2)\beta_2\big) - \mathrm{Ad}\left([A_1, B_1][A_2, B_2]\right)\big(\beta_2 + \mathrm{Ad}(B_2)\alpha_2\big) \\ &+ \dots \\ &= \sum_{i=1}^{g} \mathrm{Ad}\left(\prod_{j=1}^{i-1}[A_j, B_j]\right)\big(\alpha_i + \mathrm{Ad}(A_i)\beta_i\big) - \mathrm{Ad}\left(\prod_{j=1}^{i}[A_j, B_j]\right)\big(\beta_i + \mathrm{Ad}(B_i)\alpha_i\big) \end{aligned} \tag{1.4.3}$$

vanishes, compare [Lab13, Prop. 5.3.12]. Once again, we identified $T_{A_i}G \cong \mathfrak{g}$ and $T_{B_i}G \cong \mathfrak{g}$ via left translation.

To see the correspondence between this description of the Zariski tangent space and that of Corollary 1.4.5, we proceed as follows. First, if one defines $v\colon \pi_{g,0} \to \mathfrak{g}$ by $v(a_i) := \alpha_i$ and $v(b_i) := \beta_i$ for $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ that satisfy (1.4.3), and extend to $\pi_{g,0}$ using $v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y)$, then v defines an element of $Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$. Indeed, it is sufficient to check that $v\big(\prod[a_i, b_i]\big) = 0$. If one develops $v\big(\prod[a_i, b_i]\big)$ using $v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y)$ and $v([x, y]) = v(xy) - \mathrm{Ad}(\phi([x, y]))v(yx)$, then one gets that $v\big(\prod[a_i, b_i]\big) = 0$ is equivalent to (1.4.3) vanishing.

Conversely, given $v \in Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$, then $(v(a_1), \dots, v(a_g), v(b_1), \dots, v(b_g))$ satisfies 1.4.3 by the same argument as above.

1.5 Smooth points

Smooth points of analytic varieties in $\mathbb{R}^n$ are defined as follows.

Definition 1.5.1 (Smooth points)**.** A point x of an analytic variety $X \subset \mathbb{R}^n$ is a *smooth point* if there is an open neighbourhood $U \subset X$ of x such that U is an embedded submanifold of $\mathbb{R}^n$.

Using the Implicit Function Theorem, we can reformulate the condition and say that x is a smooth point of X if and only if the rank of the Jacobi matrix (1.4.1) at x is maximal. By the Rank-Nullity Theorem, this happens if and only if the dimension of the Zariski tangent space to X at x is minimal. If every point of an analytic variety is smooth, then it is an analytic manifold.

In the context of representation varieties, we will use the characterization of smooth points as the ones that minimize the dimension of the Zariski tangent space. For instance, if Γ is a free group, then $\mathrm{Hom}(\Gamma, G)$ is an analytic manifold because of the absence of relations (recall from Lemma 1.2.2 that representation varieties are analytic varieties).

Lemma 1.5.2. *The set of smooth points of* $\mathrm{Hom}(\Gamma, G)$ *is invariant under the* $\mathrm{Inn}(G)$*-action.*

Proof. The action of G on itself by conjugation is analytic. Therefore, it preserves smooth neighbourhoods of points inside $\mathrm{Hom}(\Gamma, G)$. We can give an alternative argument by observing that the Zariski tangent spaces at ϕ and $g\phi g^{-1}$ are isomorphic as Γ-modules, and hence have the same dimension. The isomorphism is given by

$$\begin{aligned} Z^1(\Gamma, \mathfrak{g}_\phi) &\to Z^1(\Gamma, \mathfrak{g}_{g\phi g^{-1}}) \\ v &\mapsto \mathrm{Ad}(g)v. \end{aligned}$$

□

In the case that $\Gamma = \pi_{g,0}$ is a closed surface group and G is quadrable, it is possible to describe the smooth points of the representation variety explicitly.

Proposition 1.5.3 ([Gol84])**.** *Let G be a quadrable Lie group. The smooth points of* $\mathrm{Hom}(\pi_{g,0}, G)$ *are those representations ϕ satisfying*

$$\dim Z(G) = \dim Z(\phi),$$

where $Z(G)$ denotes the centre of G and $Z(\phi)$ is the centralizer of $\phi(\pi_{g,0})$ inside G (the dimensions are to be understood in terms of manifolds here).

Proof. We compute the dimension of the Zariski tangent space to $\mathrm{Hom}(\pi_{g,0}, G)$ at ϕ. We use the identification with $Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$ provided by Corollary 1.4.5. Recall that the group cohomology of $\pi_{g,0}$ with coefficients in $\mathfrak{g}_\phi$ is isomorphic to the de Rham cohomology of the surface $\Sigma_{g,0}$ with coefficients in the flat vector bundle E_ϕ associated to $\mathfrak{g}_\phi$ (i.e. the adjoint bundle of the principal G-bundle $(\widetilde{\Sigma}_{g,0} \times G)/\pi_{g,0}$ built from ϕ, see [Gol84] for more details):

$$H^*(\pi_{g,0}, \mathfrak{g}_\phi) \cong H^*_{dR}(\Sigma_{g,0}, E_\phi).$$

In particular, it vanishes in degrees larger than 2.

Goldman observed that the quantity

$$\dim H^0(\pi_{g,0},\mathfrak{g}_\phi) - \dim H^1(\pi_{g,0},\mathfrak{g}_\phi) + \dim H^2(\pi_{g,0},\mathfrak{g}_\phi) \tag{1.5.1}$$

is independent of ϕ. Indeed, using that the space of cochains $C^*(\Sigma_{g,0}, E_\phi)$ in the de Rham complex is finite-dimensional in every degree, we conclude that (1.5.1) is equal to the alternating sum of the dimensions of the spaces of cochains in the de Rham complex. The latter is independent of ϕ, because the structure of $\pi_{g,0}$-module of $\mathfrak{g}_\phi$ only intervenes in the differential, see the definition of the bar resolution (B.2). If ϕ is the trivial representation, then $\mathfrak{g}_\phi$ is the trivial $\pi_{g,0}$-module and (1.5.1) is equal to the Euler characteristic of $\Sigma_{g,0}$ times the dimension of G. We conclude

$$\dim H^1(\pi_{g,0},\mathfrak{g}_\phi) = (2g-2)\dim G + \dim H^0(\pi_{g,0},\mathfrak{g}_\phi) + \dim H^2(\pi_{g,0},\mathfrak{g}_\phi).$$

Poincaré duality (see Appendix B.7) implies $H^2(\pi_{g,0},\mathfrak{g}_\phi) \cong H^0(\pi_{g,0},\mathfrak{g}_\phi^*)^*$. The existence of a non-degenerate, Ad-invariant, symmetric, bilinear form on $\mathfrak{g}$ implies that $\mathfrak{g}_\phi \cong \mathfrak{g}_\phi^*$ as $\pi_{g,0}$-modules. Hence, $\dim H^0(\pi_{g,0},\mathfrak{g}_\phi) = \dim H^2(\pi_{g,0},\mathfrak{g}_\phi)$. It is easy to see that $H^0(\pi_{g,0},\mathfrak{g}_\phi)$ is the space of $\mathrm{Ad}(\phi)$-invariant elements of $\mathfrak{g}$, namely $\mathfrak{z}(\phi)$. Hence

$$\dim H^1(\pi_{g,0},\mathfrak{g}_\phi) = (2g-2)\dim G + 2\dim Z(\phi).$$

Recall from (1.4.2) that the dimension of $B^1(\pi_{g,0},\mathfrak{g}_\phi)$ is equal to $\dim G - \dim Z(\phi)$. Finally, we obtain

$$\dim Z^1(\pi_{g,0},\mathfrak{g}_\phi) = (2g-1)\dim G + \dim Z(\phi).$$

Since $Z(G) \subset Z(\phi)$, it holds that $\dim Z(G) \leqslant \dim Z(\phi)$, and we conclude that ϕ minimizes the dimension of its Zariski tangent space if and only if $\dim Z(G) = \dim Z(\phi)$. □

Alternative proof. Instead of using group cohomology (and the embedding of the representation variety in G^Γ), one can alternatively compute the dimension of the Zariski tangent space at a representation ϕ from the embedding $\mathrm{Hom}(\pi_{g,0}, G) \subset G^{2g}$, compare [Lab13, Prop. 5.3.12]. The infinitesimal kernel of the unique relation of a closed surface group is described by (1.4.3), where $A_i = \phi(a_i)$ and $B_i = \phi(b_i)$.

Consider the orthogonal complement V in $\mathfrak{g}$, with respect to the Ad-invariant pairing B coming from the quadrability of G, of the image of the map $\mu\colon \mathfrak{g}^{2g} \to \mathfrak{g}$ defined by (1.4.3). A simple computation leads to

$$\begin{aligned}\mu(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g) = &\sum_{i=1}^{g}\left(\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\right)(\alpha_i - \mathrm{Ad}(A_iB_iA_i^{-1})\alpha_i)\\ &- \sum_{i=1}^{g}\left(\prod_{j\leqslant i}\mathrm{Ad}\left([A_j,B_j]\right)\right)(\beta_i - \mathrm{Ad}(B_iA_iB_i^{-1})\beta_i).\end{aligned}$$

The orthogonal complement of the Lie algebra of the centralizer $Z(g)$ of any element $g \in G$ is equal to the image of the map $\mathfrak{g} \to \mathfrak{g}$ given by $\xi \mapsto \xi - \mathrm{Ad}(g)\xi$. Therefore, using the general fact that

$Z(ghg^{-1}) = gZ(h)g^{-1}$ for any $g, h \in G$, we obtain that V must contain the Lie algebra of

$$\begin{aligned}
&\bigcap_{i=1}^{g} \prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right)\left(Z(A_i B_i A_i^{-1}) \cap Z(A_i B_i A_i B_i^{-1} A_i^{-1})\right) \\
&= \bigcap_{i=1}^{g} \prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right) \mathrm{Ad}(A_i B_i)\left(Z(B_i) \cap Z(A_i)\right) \\
&= \bigcap_{i=1}^{g} \prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right)\left(Z(B_i) \cap Z(A_i)\right) \\
&= \bigcap_{i=1}^{g} \left(Z(A_i) \cap Z(B_i)\right).
\end{aligned}$$

Hence, $\mathfrak{z}(\phi) \subset V$. The reverse inclusion is obvious. Using the Rank-Nullity Theorem, we conclude, as before, that the dimension of the Zariski tangent space at the representation ϕ is

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\phi) = \dim \mathrm{Ker}(\mu) = (2g-1)\dim G + \dim Z(\phi). \qquad \square$$

Proposition 1.5.3 applies to closed surface groups. In Proposition 4.2.5 below, we will discuss an analogous description of smooth points for fundamental groups of punctured surfaces.

Chapter 2

The action by conjugation

In this section, we elaborate on the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ by post-composition. We sometimes refer to this action as the *the conjugation action of G on the representation variety.*

2.1 Freeness

The action of $\mathrm{Inn}(G) \cong G/Z(G)$ on $\mathrm{Hom}(\Gamma, G)$ is never free, since the trivial representation is always a global fixed point. It is easy to see that the stabilizer of a representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is $Z(\phi)/Z(G)$. In particular

Lemma 2.1.1. *The $\mathrm{Inn}(G)$-action is free on the $\mathrm{Inn}(G)$-invariant subset that consists of all the representations ϕ such that*

$$Z(G) = Z(\phi).$$

There is a neat characterization of the points where the action is locally free. Recall that the action of a topological group on a set X is *locally free at $x \in X$* if the stabilizer of x is discrete.

Proposition 2.1.2 ([Gol84])**.** *The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is locally free at ϕ if and only if*

$$\dim Z(G) = \dim Z(\phi).$$

Proof. The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ induces, for any representation ϕ, a surjective linear map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$, where $\mathfrak{Inn}(G)$ denotes the Lie algebra of $\mathrm{Inn}(G)$ and $\mathcal{O}_\phi$ the $\mathrm{Inn}(G)$-orbit of ϕ. The map is given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)(\phi).$$

Observe that the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is locally free at ϕ if and only if the induced map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$ is injective. Since the map is always surjective, this is equivalent to asking that both spaces $\mathfrak{Inn}(G)$ and $T_\phi \mathcal{O}_\phi$ have the same dimension. The dimension of $\mathfrak{Inn}(G)$ is $\dim G - \dim Z(G)$ and the dimension of $T_\phi \mathcal{O}_\phi$ is $\dim G - \dim Z(\phi)$, as computed in (1.4.2). Hence, the dimensions coincide if and only if $\dim Z(G) = \dim Z(\phi)$. $\square$

Example 2.1.3 (Surface groups)**.** It is striking that the condition of Proposition 2.1.2 coincides with that of Proposition 1.5.3. This means that if $\Gamma = \pi_{g,0}$ is a closed surface group, then the smooth points of $\mathrm{Hom}(\pi_{g,0}, G)$ are precisely those where the action of $\mathrm{Inn}(G)$ is locally free.

Proposition 2.1.2 motivates the following definition.

Definition 2.1.4 (Regular representations)**.** A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is called *regular* if

$$\dim Z(G) = \dim Z(\phi).$$

We denote by $\mathrm{Hom}^{\mathrm{reg}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of regular representations. If it further holds that $Z(G) = Z(\phi)$, we say that ϕ is *very regular*. The $\mathrm{Inn}(G)$-invariant subspace of very regular representations is denoted by $\mathrm{Hom}^{\mathrm{vReg}}(\Gamma, G)$.

We will see later that if G is a reductive algebraic group, then most representations are regular, see Proposition 2.2.9.

Example 2.1.5. In the case $G = \mathrm{PSL}(2,\mathbb{R})$, the representations $\phi\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ that are not regular are of a particular kind. We use the description of centralizers in $\mathrm{PSL}(2,\mathbb{R})$ provided by Lemma A.9. It tells us that a non-regular representation is of one of the following kinds:

1. ϕ is the trivial representation.
2. The elements of $\phi(\Gamma)$ are rotations around the same point of $\mathbb{H}$ and $Z(\phi) \cong \mathrm{PSO}(2,\mathbb{R})$.
3. The elements of $\phi(\Gamma)$ fix a common geodesic in $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}_{>0}$.
4. The elements of $\phi(\Gamma)$ fix the same point in the boundary of $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}$.

As soon as the image of $\phi(\Gamma)$ contains, for instance, two elements of different nature (elliptic, hyperbolic or parabolic) or two rotations around different points, then $Z(\phi) = Z(\mathrm{PSL}(2,\mathbb{R}))$ is trivial and ϕ is regular, actually very regular.

2.2 Properness

The conjugation action of G on $\mathrm{Hom}(\Gamma, G)$ is in general not proper.

Example 2.2.1. Consider the case where $\Gamma = F_2 = \langle a, b\rangle$ is the free group on two generators and $G = \mathrm{PSL}(2,\mathbb{R})$. Let $\phi_1\colon F_2 \to \mathrm{PSL}(2,\mathbb{R})$ be the representation given by $\phi_1(a) = \mathrm{par}^+$ (see (A.6)) and $\phi_1(b)$ is the identity. Let ϕ_2 denote the trivial representation. Since the closure of the conjugacy class of any parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ contains the identity, we observe that

$$\phi_2 \in \overline{\mathcal{O}_{\phi_1}} \smallsetminus \mathcal{O}_{\phi_1} \quad \text{and} \quad \{\phi_2\} = \mathcal{O}_{\phi_2}.$$

So, the orbits $\mathcal{O}_{\phi_1}$ and $\mathcal{O}_{\phi_2}$ cannot be separated by disjoint open sets in the (topological) quotient $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{Inn}(\mathrm{PSL}(2,\mathbb{R}))$. In particular, the quotient is not Hausdorff and the conjugacy action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))$ is not proper.

Example 2.2.1 hints at the pathological behaviour of representations whose image lies in a parabolic subgroup. This is essentially a worst case scenario, as we explain below.

Definition 2.2.2 (Borel and parabolic subgroups)**.** Let G be an algebraic group. A *Borel subgroup* of G is a maximal, Zariski closed, solvable connected subgroup of G. A Zariski closed subgroup of G that contains a Borel subgroup is called a *parabolic subgroup* of G.

By definition, a Borel subgroup of G is automatically a Borel subgroup of G°. Similarly, P is a parabolic subgroup of G if and only if P° is a parabolic subgroup of G°. If G is connected, then all parabolic subgroups are connected [Mil17, Cor. 17.49].

Example 2.2.3. Let $G = \mathrm{GL}(n, \mathbb{C})$. The subgroup of upper triangular matrices is a Borel subgroup of G. More generally, the Borel subgroups of $\mathrm{GL}(n, \mathbb{C})$ are the ones that preserve a full flag in $\mathbb{C}^n$ and the parabolic subgroups are those that preserve a (partial) flag in $\mathbb{C}^n$ [Bou05, Chap. VIII, §13].

Definition 2.2.4 (Irreducible representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *irreducible* if the image of ϕ does not lie in a proper parabolic subgroup of G. We denote by $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of irreducible representations.

Observe that if $G = \mathrm{GL}(n, \mathbb{C})$, then ϕ being irreducible in the sense of Definiton 2.2.4 is equivalent to $\mathbb{C}^n$ being an irreducible Γ-module (i.e. ϕ is an irreducible representation in the classical sense). This is a consequence of Example 2.2.3.

Example 2.2.5. Let $G = \mathrm{SL}(2, \mathbb{C})$. The irreducible representations into $\mathrm{SL}(2, \mathbb{C})$ can be characterized in terms of traces:

Lemma 2.2.6. *A representation $\phi\colon \Gamma \to G$ is irreducible if and only there exists an element $\gamma \in [\Gamma, \Gamma] \subset \Gamma$ of the commutator subgroup of Γ such that $\mathrm{Tr}(\phi(\gamma)) \neq 2$.*

A proof of Lemma 2.2.6 can be found in [CS83, Lem. 1.2.1]. The argument relies on the following observation: if $A, B \in \mathrm{SL}(2, \mathbb{C})$ are two upper-triangular matrices, then their commutator $[A, B]$ is upper-triangular and has trace 2 (i.e. upper-triangular with ones on the diagonal).

Definition 2.2.7 (Irreducible subgroups)**.** A subgroup of an algebraic group G is called *irreducible* if it is not contained in a proper parabolic subgroup of G.

In particular, a representation $\phi\colon \Gamma \to G$ is irreducible if and only if its image is an irreducible subgroup of G. The centralizer of an irreducible subgroup in a reductive group G is a finite extension of $Z(G)$ [Sik12, Prop. 15] (see also [Sik12, Cor. 17]). Hence

Lemma 2.2.8. *Let G be a reductive algebraic group. Irreducible representations into G are regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

It is important to note the following

Proposition 2.2.9. *Let G be a reductive algebraic group. The subspace of irreducible representations $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ is Zariski open in the representation variety $\mathrm{Hom}(\Gamma, G)$. Moreover, if $\Gamma = \pi_{g,n}$ is a surface group, then $\mathrm{Hom}^{\mathrm{irr}}(\pi_{g,n}, G)$ is dense in a nonempty set of irreducible components of $\mathrm{Hom}(\pi_{g,n}, G)$.*

We refer the reader to [Sik12, Prop. 27 & 29] for a proof. The main result of this section says that if one restricts to irreducible representations, then the conjugation action of G becomes proper.

Theorem 2.2.10 ([JM87])**.** *Let G be a reductive algebraic group. The $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ is proper.*

We refer the reader to [JM87, Prop. 1.1] and references therein for a proof of Theorem 2.2.10. Following [JM87], we introduce the notion of good representations.

Definition 2.2.11 (Good representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *good*[1] if it is irreducible and very regular. We denote by $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of good representations.

Lemma 2.1.1 implies that the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is free and by Theorem 2.2.10 it is also proper. It is, however, not clear a priori whether good representations exist. However, one can prove the following

Lemma 2.2.12 ([JM87])**.** *Let G be a reductive algebraic group. The set of good representations $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is Zariski open in the representation variety $\mathrm{Hom}(\Gamma, G)$.*

Lemma 2.2.12 is proven in [JM87, Prop 1.3 & Lem. 1.3]. In general, $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ might not be a smooth manifold. However, it is the case for closed surface groups by Proposition 1.5.3. We conclude from Theorem 2.2.10 and Lemma 2.1.1 that

Corollary 2.2.13. *Let G be a reductive algebraic group. Let $\Gamma = \pi_{g,0}$ be a closed surface group. The space of good representations $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ is an analytic manifold of dimension $(2g-1)\dim G + \dim Z(G)$. The $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ is proper and free, and the quotient*

$$\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/\,\mathrm{Inn}(G)$$

is an analytic manifold of dimension $(2g-2)\dim G + 2\dim Z(G)$.

Note that the dimension of the quotient in Corollary 2.2.13 is always even. This observation will be relevant later in Section 4 when we discuss the symplectic nature of character varieties.

The notion of irreducible representations can be generalized to the notion of reductive representations.

Definition 2.2.14 (Linearly reductive groups)**.** An algebraic group is called *linearly reductive* if all its finite-dimensional representations are completely reducible.

Equivalently, over the fields of real or complex numbers, an algebraic group G is linearly reductive if and only if the algebraic subgroup that consists of the identity component for the Zariski topology is reductive [Mil17, Cor. 22.43].

Definition 2.2.15 (Completely reducible subgroups)**.** A subgroup of an algebraic group is called *completely reducible* if and only if its Zariski closure is linearly reductive.

[1] In [JM87] and [Sik12] a good representation is defined to be a very regular reductive representation (see Definition 2.2.16). If G is reductive, then their definition is equivalent to ours (see Lemma 2.2.18).

Definition 2.2.16 (Reductive representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *reductive* (or *completely reducible*) if $\phi(\Gamma) \subset G$ is completely reducible. We denote by $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of reductive representations.

In particular, a representation $\phi\colon \Gamma \to \mathrm{GL}(n,\mathbb{C})$ is reductive if and only if $\mathbb{C}^n$ is a completely reducible Γ-module (i.e. a direct sum of irreducible Γ-modules).

Lemma 2.2.17. *Let G be a reductive algebraic group. Irreducible representations $\phi\colon \Gamma \to G$ are reductive:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{red}}(\Gamma, G).$$

Proof. The proof relies on the observation that irreducible subgroups of reductive algebraic groups are completely reducible. This is proved in [Sik12, §3] using the notion of Levi subgroups. ☐

The converse of Lemma 2.2.17 is not true in general. However

Lemma 2.2.18. *Let G be a reductive algebraic group. A reductive representation into G is irreducible if and only if it is regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G) \cap \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

The reader is referred to [Sik12, Cor. 17] for a proof of Lemma 2.2.18. Reductive representations can be characterized as follows:

Proposition 2.2.19. *Let G be a reductive algebraic group. A representation $\phi\colon \Gamma \to G$ is reductive if and only if the the $\mathrm{Inn}(G)$-orbit $\mathcal{O}_\phi$ of ϕ is closed in $\mathrm{Hom}(\Gamma, G)$.*

A proof of Proposition 2.2.19 can be found in [Sik12, Thm. 30], based on an argument of [JM87]. An immediate consequence of Proposition 2.2.19 is that the points of the topological quotient $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\mathrm{Inn}(G)$ are closed, i.e. it is a $\mathcal{T}_1$ space[2].

Proposition 2.2.20 ([RS90])**.** *Let G be a reductive algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\mathrm{Inn}(G)$$

is Hausdorff.

The reader is referred to [RS90, §7.3] and references therein for a proof of Proposition 2.2.20. Some authors favour the notion of Zariski dense representations over irreducible representations, see for instance [Lab13], [Mon16].

Definition 2.2.21 (Zariski dense representations)**.** Let G be an algebraic Lie group. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is called *Zariski dense* if $\phi(\Gamma)$ is a Zariski dense subgroup of G. It is called *almost Zariski dense* if the Zariski closure of $\phi(\Gamma)$ contains G°. The $\mathrm{Inn}(G)$-invariant spaces of Zariski dense and almost Zariski dense representations are denoted $\mathrm{Hom}^{\mathrm{Zd}}(\Gamma, G)$ and $\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G)$, respectively.

[2] See Section 3 for a reminder of some notions of separability.

Recall that a subgroup H of an algebraic groups G is Zariski dense if and only if any regular function that vanishes on H also vanishes on G.

Lemma 2.2.22. *Let G be an algebraic Lie group. Almost Zariski dense representations are irreducible:*

$$\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{irr}}(\Gamma, G).$$

Proof. Let $\phi\colon \Gamma \to G$ be almost Zariski dense. By definition, the Zariski closure of $\phi(\Gamma)$ contains G°. In particular, no proper parabolic subgroups of G° can contain the identity component of the Zariski closure of $\phi(\Gamma)$. Since parabolic subgroups are by definition Zariski closed, no proper parabolic subgroup of G can contain $\phi(\Gamma)$. □

Example 2.2.23. Let $\alpha_1, \ldots, \alpha_n \in (0, 2\pi)^n$ be angles such that $\alpha_1 + \ldots + \alpha_n = 2k\pi$ for some integer k. Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group on n generators. We consider the representation $\phi\colon F_n \to \mathrm{PSL}(2, \mathbb{R})$ defined by $\phi(a_i) = \mathrm{rot}_{\alpha_i}$ (see (A.2)). The representation ϕ is not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{R})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{R})$. However, ϕ is irreducible as one can check that $\phi(\Gamma)$ has no fixed point in $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$. Consider now the representation $\overline{\phi}$ defined as the composition of ϕ with the inclusion $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{C})$. Observe that $\overline{\phi}\colon F_n \to \mathrm{PSL}(2, \mathbb{C})$ is reducible since it fixes $[1 : i] \in \mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^\times$, but it is still not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{C})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 2.2.24. *Let G be an algebraic group such that $Z(G) = Z(G^\circ)$. If $\phi \in \mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G)$, then ϕ is very regular, i.e.*

$$Z(G) = Z(\phi).$$

In particular, almost Zariski dense representations are good:

$$\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{good}}(\Gamma, G).$$

Proof. The argument is taken from [Lab13, §5.3]. Denote by $Z(Z(\phi))$ the centralizer of $Z(\phi) = Z(\phi(\Gamma))$ in G. It is a Zariski closed subgroup of G that contains $\phi(\Gamma)$. Hence, by almost Zariski density of $\phi(\Gamma)$, it holds $G^\circ \subset Z(Z(\phi))$ and thus $Z(\phi) \subset Z(G^\circ)$. Since we assumed $Z(G^\circ) = Z(G)$, we conclude that $Z(G) = Z(\phi)$. It now follows from 2.2.22 that almost Zarsiki dense representations are good. □

It follows from Theorem 2.2.10 and Lemma 2.2.22 that, for a reductive algebraic group G (hence connected) and $\Gamma = \pi_{g,0}$ a closed surface group, the $\mathrm{Inn}(G)$-action on the subspace of Zariski dense representations is free and proper, compare [Lab13, Thm. 5.2.6] and [Mon16, Lem. 2.10]. It is interesting to note that the resulting quotient, at least in the case when $Z(G)$ is finite, has the same dimension as the quotient from Corollary 2.2.13.

By way of conclusion, we provide the reader with a Venn diagram that illustrates the different relations of inclusion between the various notions of representations introduced in this section, see Figure 2.1.

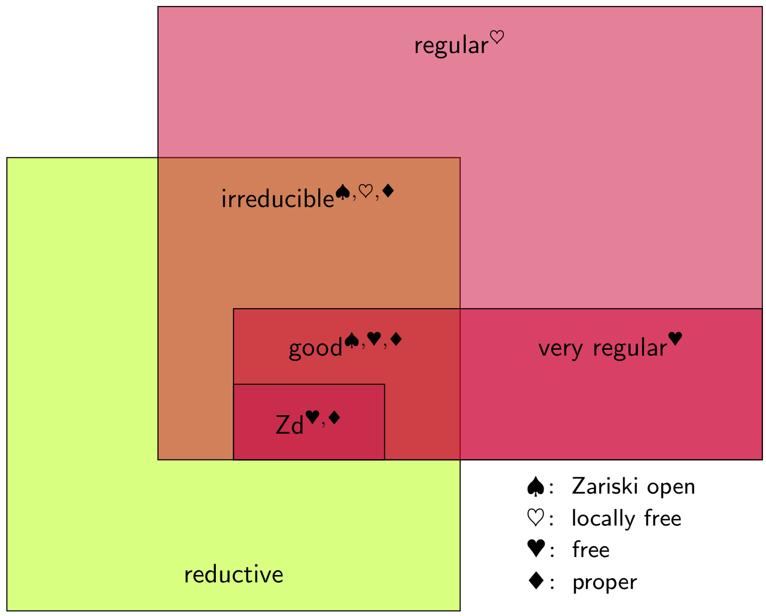


Figure 2.1: We assume for simplicity that G is a reductive algebraic group (hence connected). The two largest families of representations are the regular and the reductive ones. Their intersection is the set of irreducible representations. A representation that is irreducible and very regular is called good. Zariski dense representations are good.

2.3 Invariant functions

The real- or complex-valued functions of $\mathrm{Hom}(\Gamma, G)$ that are invariant under the conjugation action of G are called *invariant functions* of the representation variety. We consider the case where G is an algebraic group over $\mathbb{C}$. The algebra of regular functions on the variety $\mathrm{Hom}(\Gamma, G)$, a.k.a. its coordinate ring, is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of invariant functions is denoted by

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G.$$

In this section, we will only consider the case of a linear algebraic group $G \subset \mathrm{GL}(m, \mathbb{C})$. The main example of invariant functions are the so-called trace functions (recall that $\mathrm{Tr}\colon \mathrm{GL}(m, \mathbb{C}) \to \mathbb{C}$ is a conjugacy invariant).

Definition 2.3.1 (Trace functions)**.** Let $\gamma \in \Gamma$. The function

$$\begin{aligned} \mathrm{Tr}_\gamma \colon \mathrm{Hom}(\Gamma, G) &\to \mathbb{C} \\ \phi &\mapsto \mathrm{Tr}(\phi(\gamma)). \end{aligned}$$

is called the *trace function of* γ. We denote by $\mathcal{T}(\Gamma, G)$ the subalgebra of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ generated by trace functions.

In most cases, as for instance when G is one of the classical complex Lie groups, invariant functions of the representation variety are generated by trace functions. In other words, $\mathcal{T}(\Gamma, G) = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. This is a consequence of Procesi's Theorem (see Theorem 2.3.3 below) on invariants of matrices.

Remark 2.3.2. Nagata's Theorem implies that, if G is a reductive algebraic group, then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated, see for instance [Dol03, Thm. 3.3].

Let $\mathbb{K}$ denote either the field of real or complex numbers. We denote by $M_m(\mathbb{K})$ the algebra of $m \times m$ matrices with coefficients in $\mathbb{K}$. Let $M_m(\mathbb{K})^n = M_m(\mathbb{K}) \times \ldots \times M_m(\mathbb{K})$ and $\mathbb{K}[M_m(\mathbb{K})^n]$ be the algebra of polynomial functions in n matrix variables $\xi_k = (x^k_{i,j})_{i,j=1,\ldots,m}$. The group $\mathrm{GL}(m, \mathbb{K})$ acts diagonally on $M_m(\mathbb{K})^n$ by conjugation. For any subgroup $G \subset \mathrm{GL}(m, \mathbb{K})$, the subalgebra of $\mathbb{K}[M_m(\mathbb{K})^n]$ that consists of G-invariant polynomials is denoted $\mathbb{K}[M_m(\mathbb{K})^n]^G$.

Theorem 2.3.3 ([Pro76])**.** *The following hold:*

- *If $G \in \{\mathrm{GL}(m, \mathbb{K}), \mathrm{SL}(m, \mathbb{K})\}$, then $\mathbb{K}[M_m(\mathbb{K})^n]^G$ is finitely generated by trace polynomials $\mathrm{Tr}(W)$, where W is a reduced word in $\xi_1, \ldots, \xi_n$ of length at most $2^m - 1$.*
- *If $G \in \{\mathrm{O}(m, \mathbb{K}), \mathrm{SO}(m, \mathbb{K})\}$, then $\mathbb{K}[M_m(\mathbb{K})^n]^G$ is finitely generated by trace polynomials $\mathrm{Tr}(W)$, where W is a reduced word of length at most $2^m - 1$ in $\xi_1, \ldots, \xi_n$ and their orthogonal transposes*[3]*.*
- *If $G = \mathrm{Sp}(2m, \mathbb{K})$, then $\mathbb{K}[M_{2m}(\mathbb{K})^n]^G$ is finitely generated by trace polynomials $\mathrm{Tr}(W)$, where W is a reduced word of length at most $2^m - 1$ in $\xi_1, \ldots, \xi_n$ and their symplectic transposes*[4]*.*

The reader is referred to [Pro76] for the proof of Theorem 2.3.3, see also [DCP17].

Back to the context of representation varieties: Assume that Γ admits a generating family $(\gamma_1, \ldots, \gamma_n)$, then the embedding $\imath\colon \mathrm{Hom}(\Gamma, G) \subset G^n$ induces a surjective morphism

$$\imath^*\colon \mathbb{C}[G^n] \twoheadrightarrow \mathbb{C}[\mathrm{Hom}(\Gamma, G)]. \tag{2.3.1}$$

The morphism $\imath^*$ maps invariant functions to invariant functions and thus restricts to a morphism

$$(\imath^*)^G\colon \mathbb{C}[G^n]^G \to \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G. \tag{2.3.2}$$

If we further assume G to be reductive, then $(\imath^*)^G$ is surjective. This is a consequence of the existence of Reynolds operators, see [Sik13, Rem. 25] or [Hos15, Cor. 4.23]. The morphism $(\imath^*)^G$ maps trace functions to trace functions in the following sense.

Lemma 2.3.4. *Let W be a reduced word in the matrices variables $\xi_1, \ldots, \xi_n$. It holds that*

$$(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\ldots,\gamma_n)}\,.$$

Proof. The word W induces a word map $W\colon G^n \to G$. The trace function $\mathrm{Tr}(W)\colon G^n \to \mathbb{C}$ sends $(g_1, \ldots, g_n)$ to $\mathrm{Tr}(W(g_1, \ldots, g_n))$. The image $(\imath^*)^G(\mathrm{Tr}(W))$ is the invariant function $\mathrm{Hom}(\Gamma, G) \to \mathbb{C}$ given by $\phi \mapsto \mathrm{Tr}(W(\phi(\gamma_1), \ldots, \phi(\gamma_n)))$. Because ϕ is a group homomorphism, it holds that

[3]The *orthogonal transpose* of a matrix is the inverse of its transpose. The orthogonal group $\mathrm{O}(m, \mathbb{K})$ consists precisely of the matrices that are equal to their orthogonal transposes.

[4]The *symplectic transpose* of a matrix $A \in M_{2m}(\mathbb{K})$ is the matrix JA^tJ, where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and I_m is the $m \times m$ identity matrix. The symplectic group $\mathrm{Sp}(2m, \mathbb{K})$ consists precisely of the matrices that are equal to their symplectic transposes.

$\mathrm{Tr}(W(\phi(\gamma_1), \ldots, \phi(\gamma_n))) = \mathrm{Tr}(\phi(W(\gamma_1, \ldots, \gamma_n))$, where we now think of W as a function $W : \Gamma^n \to \Gamma$. We conclude that $(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\ldots,\gamma_n)}$. $\square$

Lemma 2.3.5. *Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a reductive linear algebraic group. If the algebra $\mathbb{C}[G^n]^G$ is generated by trace functions, then*

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G = \mathcal{T}(\Gamma, G).$$

Proof. If G is reductive, then $(\imath^*)^G$ is surjective and so $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Moreover, $(\imath^*)^G$ maps trace functions to trace functions, thus, if $\mathbb{C}[G^n]^G$ is generated by trace functions, then it holds $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathcal{T}(\Gamma, G)$. $\square$

Lemma 2.3.6. *Let G be one of the reductive groups $\mathrm{GL}(m, \mathbb{C})$ or $\mathrm{SL}(m, \mathbb{C})$ with $m \geqslant 2$, $\mathrm{O}(m, \mathbb{C})^\circ$ or $\mathrm{SO}(m, \mathbb{C})$ with $m \geqslant 3$, or $\mathrm{Sp}(2m, \mathbb{C})$. Then $\mathbb{C}[G^n]^G$ is generated by trace functions.*

Proof. The inclusion $G \subset M_m(\mathbb{C})$ induces a surjective morphism $\mathbb{C}[M_m(\mathbb{C})^n]^G \twoheadrightarrow \mathbb{C}[G^n]^G$. Theorem 2.3.3 says that $\mathbb{C}[M_m(\mathbb{C})^n]^G$ is generated by trace of words of matrices and their transposes. In particular, a similar argument as in the proof of Lemma 2.3.5 implies that $\mathbb{C}[G^n]^G$ is generated by traces of words. We used here that the inverse transpose and the symplectic transpose of any matrix in $\mathrm{O}(m, \mathbb{C})$ and $\mathrm{Sp}(2m, \mathbb{C})$, respectively, is the matrix itself. $\square$

We conclude

Corollary 2.3.7. *Let G be one of the reductive groups $\mathrm{GL}(m, \mathbb{C})$ or $\mathrm{SL}(m, \mathbb{C})$ with $m \geqslant 2$, $\mathrm{O}(m, \mathbb{C})^\circ$ or $\mathrm{SO}(m, \mathbb{C})$ with $m \geqslant 3$, or $\mathrm{Sp}(2m, \mathbb{C})$. Then*

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G = \mathcal{T}(\Gamma, G).$$

Example 2.3.8. Let $G = \mathrm{SL}(2, \mathbb{C})$. Corollary 2.3.7 says that the algebra of invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})}$ is generated by Tr_γ for $\gamma \in \Gamma$. The trace formula $\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1})$ for 2×2 matrices gives the relation

$$\mathrm{Tr}_{\gamma_1} \mathrm{Tr}_{\gamma_2} = \mathrm{Tr}_{\gamma_1\gamma_2} + \mathrm{Tr}_{\gamma_1\gamma_2^{-1}}.$$

It is folklore knowledge (see [MS21, §1]) that the trace formula, together with the relation $\mathrm{Tr}_1 = 2$, is a complete set of relations. In other words, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(2, \mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})} \cong \mathbb{C}[X_\gamma : \gamma \in \Gamma] \Big/ \left(X_1 - 2, X_{\gamma_1}X_{\gamma_2} - X_{\gamma_1\gamma_2} - X_{\gamma_1\gamma_2^{-1}}\right).$$

2.4 Characters

A character is the analogue of a trace function where a representation is now fixed and $\gamma \in \Gamma$ is the variable. We assume again that $G \subset \mathrm{GL}(m, \mathbb{C})$ is a linear algebraic group.

Definition 2.4.1 (Characters)**.** The *character* of a representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is the function

$$\begin{aligned} \chi_\phi \colon \Gamma &\to \mathbb{C} \\ \gamma &\mapsto \mathrm{Tr}(\phi(\gamma)). \end{aligned}$$

In other words, $\chi_\phi(\gamma) = \mathrm{Tr}_\gamma(\phi)$. We denote by $\chi(\Gamma, G) \subset \mathbb{C}^\Gamma$ the set of all characters of representations in $\mathrm{Hom}(\Gamma, G)$ equipped with the subspace topology inherited from the compact-open topology on $\mathbb{C}^\Gamma$.

Note that $\chi(\Gamma, G) \subset \mathbb{C}^\Gamma$ is automatically a Hausdorff space because $\mathbb{C}^\Gamma$ is a Hausdorff space.

Theorem 2.4.2 ([CS83])**.** *The space $\chi(\Gamma, G) \subset \mathbb{C}^\Gamma$ is a closed algebraic variety for $G = \mathrm{SL}(2, \mathbb{C})$.*

We refer the reader to [CS83, Cor. 1.4.5] for a proof of Theorem 2.4.2. The natural projection

$$\mathrm{Hom}(\Gamma, G) \to \chi(\Gamma, G)$$

factors through the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. A character does not necessarily determine a unique conjugacy class of representations. For instance, the two representations of Example 2.2.1 are not conjugate but determine the same character. However, the following is true.

Proposition 2.4.3. *Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a linear algebraic group. Conjugacy classes of irreducible representations are determined by their characters.*

Culler–Shalen provide a proof of Proposition 2.4.3 in [CS83, Prop. 1.5.2] for the case $G = \mathrm{SL}(2, \mathbb{C})$ and claim that the result still holds when $\mathrm{SL}(2, \mathbb{C})$ is replaced by $\mathrm{GL}(m, \mathbb{C})$. The analogous result for almost Zariski dense representations can be found in [Lab13, Cor. 5.3.7].

Chapter 3

Character varieties

The previous sections highlighted the relevance of the quotient space $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. However, it was also explained that there is no reason to expect that this quotient has any nice structure, since the action of G by conjugation on the representation variety is non-free and non-proper in general. The goal of this section is to construct an alternative space, with a nicer structure than the topological quotient $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ and with a projection from $\mathrm{Hom}(\Gamma, G)$ that factors through $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. The specification is to construct the largest possible space, while ensuring some regularity such as being Hausdorff or being a variety or manifold. The resulting space will be called a *character variety* of the finitely generated group Γ and the Lie group G. Several constructions explained below lead to richer structures but require more assumptions on the Lie group G.

We start by recalling the definitions of two separability properties. A topological space X is said to be

- $\mathcal{T}_1$ if for any pair of distinct points in X, each point lies in an open set that does not contain the other, or, equivalently, X is $\mathcal{T}_1$ if the points of X are closed,
- $\mathcal{T}_2$ or *Hausdorff* if for any pair of distinct points in X, there are two disjoint open sets such that each contains one of the two points.

Note that the quotient $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))/\operatorname{Inn}(\mathrm{PSL}(2,\mathbb{R}))$ of Example 2.2.1 is not only non-Hausdorff, but is also not $\mathcal{T}_1$. Indeed, the closure of the orbit of ϕ_1 always contains the orbit of ϕ_2.

3.1 Hausdorff quotient

The first approach consists in considering the Hausdorffization the topological quotient. The Hausdorffization of a topological space X is basically the largest Hausdorff quotient of X.

Definition 3.1.1 (Hausdorffization)**.** Consider the equivalence relation on X given by $x \sim y$ if and only if $x \approx y$ for all equivalence relations $\approx$ on X such that $X/\approx$ is Hausdorff (such a relation $\approx$ always exists, as one can identify all the points of X). The quotient

$$\mathrm{Haus}(X) := X/\sim$$

is the *Hausdorffization of* X.

Lemma 3.1.2. *The space* $\mathrm{Haus}(X)$ *is a Hausdorff topological space. Moreover, the space* $\mathrm{Haus}(X)$ *has the following universal property: If* Y *is a Hausdorff topological space, then any continuous surjective map* $X \to Y$ *factors uniquely through the projection* $X \to \mathrm{Haus}(X)$.

Proof. First we prove that $\mathrm{Haus}(X)$ is a Hausdorff space. Let $x, y \in X$ be two points with $x \nsim y$. By definition, there exists an equivalence relation $\approx$ on X with Hausdorff quotient such that $x \not\approx y$. Since the projections of x and y in $X/\approx$ are separable and the map $X/\sim \to X/\approx$ is continuous, the projections of x and y are also separable in $X/\sim$.

Let now Y be a Hausdorff space and $f\colon X \to Y$ be a continuous surjection. Define an equivalence relation on X by $x \approx y$ if and only if $f(x) = f(y)$. The quotient $X/\approx$ is homeomorphic to the Hausdorff space Y. This implies the existence of a continuous surjective map $\mathrm{Haus}(X) \to Y$ such that f is the composition $X \to \mathrm{Haus}(X) \to Y$. The factoring map is uniquely determined by f. □

Corollary 3.1.3. *If* x *and* y *are two points of* X *such that* $\overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$, *then* $x \sim y$.

Proof. Since $\mathrm{Haus}(X)$ is Hausdorff, its points are closed. In particular, the conjugacy classes for the relation $\sim$ are closed subsets of X. If we assume that $x \nsim y$, then the conjugacy classes of x and y are disjoint closed subsets of X. This implies that the closures of $\{x\}$ and $\{y\}$ are disjoint. □

Definition 3.1.4 (Hausdorff character variety)**.** The *Hausdorff character variety* of a finitely generated group Γ and a Lie group G is the Hausdorffization of the topological quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ and is denoted

$$\mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G) := \mathrm{Haus}\left(\mathrm{Hom}(\Gamma, G) \big/ \mathrm{Inn}(G)\right).$$

The construction of character varieties by Hausdorff quotients has the advantage to work in a broad sense (it could even be defined for topological groups G). It is the approach favoured in [Mon16], for instance.

3.2 $\mathcal{T}_1$ quotient

An alternative to the Hausdorff quotient is the $\mathcal{T}_1$ quotient used in [RS90, §7]. Let G be a topological group acting on a space X. For any $x \in X$, we denote the G-orbit of x by $\mathcal{O}_x$. We make the following crucial assumption:

$$\forall x \in X, \quad \overline{\mathcal{O}_x} \subset X \text{ contains a unique closed } G\text{-orbit.} \tag{3.2.1}$$

Let $X /\!\!/ G$ denote the set of closed orbits for the action of G on X and define

$$\pi\colon X \to X /\!\!/ G$$

to be the map that sends x to the unique closed orbit contained in $\overline{\mathcal{O}_x}$. A topology on $X /\!\!/ G$ is defined by declaring π to be a quotient map, i.e $Z \subset X /\!\!/ G$ is closed if and only if $\pi^{-1}(Z) \subset X$ is closed. Define a relation on X by

$$x \approx y \quad \Leftrightarrow \quad \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing.$$

Lemma 3.2.1. *Under the assumption* (3.2.1)*, the relation $\approx$ is an equivalence relation and $X /\!/ G$ is homeomorphic to the quotient $X/\approx$.*

Proof. The relation $\approx$ is obviously symmetric and reflexive. We prove that it is also transitive. Assume that $x \approx y$ and $y \approx z$. In particular, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is nonempty and thus contains an element w. Since $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is closed and G-invariant, it holds $\overline{O_w} \subset \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$. We conclude that $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ contains a unique closed orbit which is the one contained in $\overline{O_w}$. Similarly, $\overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains a unique closed orbit. By uniqueness of the closed orbit contained in $\overline{O_y}$, the two must coincide. Hence, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains $\overline{O_w}$ and is therefore nonempty. This shows that $x \approx z$.

To see that $X /\!/ G \cong X/\approx$, observe that, by the above argument, $\pi(x) = \pi(y)$ if and only if $x \approx y$. Both are quotients of X and therefore homeomorphic. $\square$

Lemma 3.2.2. *The space $X /\!/ G$ has the following universal property: For every $\mathcal{T}_1$ space Y, any continuous map $X \to Y$ that is constant on G-orbits factors uniquely through $\pi\colon X \to X /\!/ G$.*

Proof. Let Y be $\mathcal{T}_1$ with a continuous map $f\colon X \to Y$ that is constant on G-orbits. Let $x \in X$. We want to prove that f is constant on $\overline{O_x}$. Let $y = f(x)$. Since Y is $\mathcal{T}_1$, the singleton $\{y\} \subset Y$ is closed and so is $f^{-1}(y)$. Therefore, $\overline{O_x} \subset f^{-1}(y)$ and f is constant on $\overline{O_x}$. This shows that $f\colon X \to Y$ factors through $X /\!/ G$. The factoring map $\overline{f}\colon X /\!/ G \to Y$ is continuous and uniquely determined by f. $\square$

In the case that $X /\!/ G$ is a $\mathcal{T}_1$ space, then Lemma 3.2.2 says that $X /\!/ G$ is the largest $\mathcal{T}_1$ quotient of X. There is a relation between $X /\!/ G$ and the Hausdorffization of the topological quotient X/G. Namely

Lemma 3.2.3. *There is a natural surjective continuous map*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, two heads, "\pi"] & X/G \arrow[d, two heads] \\ X /\!/ G \arrow[r, dashed, two heads, "\exists"] & \mathrm{Haus}(X/G) \end{tikzcd}$$

Proof. Let x and y be two points of X. Lemma 3.2.1 says that if $\pi(x) = \pi(y)$, then $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing$. This means the closures of $\mathcal{O}_x$ and $\mathcal{O}_y$, seen as singletons in X/G, have a nonempty intersection. By Corollary 3.1.3, we conclude that x and y project to the same point in $\mathrm{Haus}(X/G)$. $\square$

Corollary 3.2.4. *If $X /\!/ G$ is Hausdorff, then it is homeomorphic to the Hausdorffization of X/G.*

Definition 3.2.5 ($\mathcal{T}_1$ character variety)**.** If the conjugation action of G on the representation variety $\mathrm{Hom}(\Gamma, G)$ satisfies property (3.2.1), we define the $\mathcal{T}_1$ *character variety* of Γ and G to be

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) := \mathrm{Hom}(\Gamma, G) /\!/ \mathrm{Inn}(G).$$

Note that the $\mathcal{T}_1$ character variety of Γ and G might not be a $\mathcal{T}_1$ space, but always lies over any $\mathcal{T}_1$ quotient of $\mathrm{Hom}(\Gamma, G)$ by Lemma 3.2.2. In particular, by Lemma 3.2.3, there is a surjection

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \twoheadrightarrow \mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G)$$

which is a homeomorphism when $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ is Hausdorff.

3.3 GIT quotient

In this section, we sketch a construction of character variety in the case that G is a complex reductive algebraic group. It is based on geometric invariant theory (GIT). The reader may consult [Sik12], [Dre04, §2] or [Lou15, §B.5] for more details.

If G is a complex algebraic group then the representation variety $\mathrm{Hom}(\Gamma, G)$ is an algebraic variety by Lemma 1.2.3. Recall that the algebra of regular functions of $\mathrm{Hom}(\Gamma, G)$ is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of G-invariant functions is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Nagata's theorem implies that $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated, see Remark 2.3.2. In particular, there is an algebraic variety denoted $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ whose algebra of polynomial functions is $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. This variety is also known as the *GIT quotient* of $\mathrm{Hom}(\Gamma, G)$.

Definition 3.3.1 (GIT character variety)**.** The *GIT character variety* of a finitely generated group Γ and a complex reductive algebraic group G is defined to be

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) := \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

The GIT character variety has by definition the structure of an algebraic variety and is, in particular, a Hausdorff topological space with the standard topology. The inclusion $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ induces a surjective morphism of algebraic varieties

$$p\colon \mathrm{Hom}(\Gamma, G) \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

We recall here some general properties of GIT quotients and refer the reader to [Dre04, §2] and [Lou15, §B.5], and references therein for proofs.

Lemma 3.3.2. *The GIT quotient* $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *has the following universal property: for every algebraic variety Y, any morphism* $\mathrm{Hom}(\Gamma, G) \to Y$ *that is constant on G-orbits factors uniquely through* $p\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$.

Lemma 3.3.3. *The GIT quotient* $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *satisfies the following properties:*

1. *For two representations $\phi_1, \phi_2 \in \mathrm{Hom}(\Gamma, G)$, it holds that*

$$p(\phi_1) = p(\phi_2) \quad \Leftrightarrow \quad \overline{\mathcal{O}_{\phi_1}} \cap \overline{\mathcal{O}_{\phi_2}} \neq \varnothing.$$

2. *Any fibre of p contains a unique closed orbit (compare (3.2.1)).*

Lemma 3.3.3, combined with Lemma 3.2.1, implies that the underlying topological structure of the GIT character variety of Γ and G coincides with the $\mathcal{T}_1$ character variety. Since the GIT character variety is a Hausdorff space, it further coincides with the Hausdorff character variety by Corollary 3.2.4:

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) \cong \mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \cong \mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G).$$

3.4 Analytic quotient

If one is interested in constructing a character variety that is an analytic manifold, one can restrict to good representations defined in Definition 2.2.11. If $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is a nonempty analytic manifold (recall from Corollary 2.2.13 that it is the case if $\Gamma = \pi_{g,0}$ is a closed surface group and G is a reductive algebraic group), then the quotient $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/\operatorname{Inn}(G)$ is an analytic manifold.

Definition 3.4.1 (Analytic character variety)**.** The *analytic character variety* of a closed surface group $\Gamma = \pi_{g,0}$ and a reductive algebraic group G is defined to be

$$\mathrm{Rep}^{\infty}(\pi_{g,0}, G) := \mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/G.$$

The topology of an analytic character variety is a Hausdorff. Hence, by Lemma 3.1.2, there is a projection from the Hausdorff character variety (which does not need to be a homeomorphism)

$$\mathrm{Rep}^{\mathcal{T}_2}(\pi_{g,0}, G) \twoheadrightarrow \mathrm{Rep}^{\infty}(\pi_{g,0}, G).$$

3.5 Variant of the GIT and analytic quotients

The GIT character variety can be described more concretely as follows.

Definition 3.5.1 (Stability of representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is

- *polystable* if $\mathcal{O}_\phi$ is closed.
- *stable* if ϕ is polystable and regular.

The $\operatorname{Inn}(G)$-invariant subspace of polystable representations is denoted $\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)$ and the subspace of stable representations is denoted $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$.

These notions are redundant if G is a reductive complex algebraic group because of the following.

Proposition 3.5.2. *Let G be a reductive complex algebraic group. Let $\phi \in \mathrm{Hom}(\Gamma, G)$ be a representation. Then*

1. *ϕ is reductive if and only if ϕ is polystable,*
2. *ϕ is irreducible if and only if ϕ is stable.*

The first assertion of Proposition 3.5.2 was already stated in Proposition 2.2.19. The second assertion is a consequence of Lemma 2.2.18.

Theorem 3.5.3. *Let G be a reductive complex algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\operatorname{Inn}(G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\operatorname{Inn}(G)$$

is homeomorphic to $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$. It contains, as an open subset, the topological quotient

$$\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)/\operatorname{Inn}(G) = \mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)/\operatorname{Inn}(G)$$

which is an orbifold whenever $Z(G)$ is finite.

Proof. Polystable representations have a closed orbit under the $\mathrm{Inn}(G)$-action by definition. So, the first statement of Lemma 3.3.3 implies that the projection $p\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ factors through an injective map

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\,\mathrm{Inn}(G) \to \mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G).$$

We can use the second statement of Lemma 3.3.3 to see that this map is also surjective.

Recall now from Proposition 2.2.9 that $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$. To prove the orbifold statement, we use that an algebraic variety over the real or the complex numbers has a finite number of connected components in the usual topology, see e.g. [DK81, Thm. 4.1]. So, if $Z(G)$ is finite, then a polystable representation $\phi\colon \Gamma \to G$ is stable if and only if $Z(\phi)$ is finite. Equivalently, ϕ is stable if and only if it has a finite stabilizer for the $\mathrm{Inn}(G)$-action. This shows that the quotient is an orbifold since the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is proper by Theorem 2.2.10. ☐

Theorem 3.5.3 says that there is a natural structure of algebraic variety on the quotient of the space of reductive representations by the $\mathrm{Inn}(G)$-action, given that G is a reductive complex algebraic group. In the case that G is a real algebraic group, we have the following

Theorem 3.5.4 ([RS90])**.** *Let G be a real algebraic group. The quotient*

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

is a real semialgebraic[1] *variety.*

Theorem 3.5.4 is proved in [RS90, Thm. 7.6].

[1] A *semialgebraic variety* is defined to be a set of points satisfying polynomial equalities and inequalities.

Chapter 4

Symplectic structure of character varieties

Throughout this section we assume that G is a quadrable Lie group. We also fix a nondegenerate, symmetric, Ad-invariant bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Goldman described in [Gol84] a natural symplectic structure on the character variety of representations of a closed surface group into a quadrable group. We remind the reader of the construction.

Assume for now that Γ is any finitely generated group. We explained in Corollary 1.4.5 that the Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at a representation ϕ can be identified with $Z^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. To define a 2-form on the representation variety $\mathrm{Hom}(\Gamma, G)$ we use the cup product in group cohomology (B.11). Combined with the pairing B, this gives a map

$$\omega\colon Z^1(\Gamma, \mathfrak{g}_\phi) \times Z^1(\Gamma, \mathfrak{g}_\phi) \xrightarrow{\smile} Z^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi) \xrightarrow{B_*} Z^2(\Gamma, \mathbb{R}). \tag{4.0.1}$$

The map ω is bilinear and anti-symmetric because the cup product is anti-symmetric in degree 1 (Lemma B.11) and B is symmetric.

Theorem 4.0.1 ([Kar92])**.** *Let $\varphi\colon Z^2(\Gamma, \mathbb{R}) \to \mathbb{R}$ be any continuous linear function that vanishes on $B^2(\Gamma, \mathbb{R})$. Then, $\varphi \circ \omega$ is a closed 2-form on $\mathrm{Hom}(\Gamma, G)$.*

The main conclusion of Theorem 4.0.1 is the statement that the form $\varphi \circ \omega$ is closed. Karshon gives an elementary proof of the closeness via direct computations in group cohomology.

The cup product of coboundaries in $B^1(\Gamma, \mathfrak{g}_\phi)$ is itself a coboundary inside $B^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi)$. This shows that the 2-form $\varphi \circ \omega$ is degenerate. Recall from Proposition 1.4.6 that the tangent space at ϕ to the G-orbit $\mathcal{O}_\phi \subset \mathrm{Hom}(\Gamma, G)$ can be identified with the 1-coboundaries $B^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. So, $\varphi \circ \omega$ is degenerate at least along the tangent directions to the G-orbit of ϕ. In general, the kernel of $\varphi \circ \omega$ might contain more degenerate directions than those which arise from $\mathcal{O}_\phi$.

Definition 4.0.2 (Goldman symplectic form)**.** In the case that the G-orbits are the only directions of degeneracy of $\varphi \circ \omega$, we denote by $\omega_{\mathcal{G}}$ the induced nondegenrate closed form on cohomology:

$$(\omega_{\mathcal{G}})_\phi\colon H^1(\Gamma, \mathfrak{g}_\phi) \times H^1(\Gamma, \mathfrak{g}_\phi) \to \mathbb{R}.$$

We say that $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$.

The index $\mathcal{G}$ refers to Goldman. We are abusing the terminology "symplectic form" here. The topological quotient $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ does not need to be a variety in general and it is abusive to say that the "Zariski tangent space" at $[\phi] \in \operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ is the quotient space $H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$. What $\omega_{\mathcal{G}}$ really is, is a 2-form on $\operatorname{Hom}(\Gamma, G)$ that is degenerate precisely along the orbits of the $\operatorname{Inn}(G)$-action.

4.1 Closed surface groups

Let $\Gamma = \pi_{g,0}$ be a closed surface group. Let $[\pi_{g,0}]$ be a generator of $H_2(\pi_{g,0}, \mathbb{Z}) \cong \mathbb{Z}$ (where $\mathbb{Z}$ is the trivial $\pi_{g,0}$-module). In other words, $[\pi_{g,0}]$ corresponds to an orientation of the surface $\Sigma_{g,0}$ under the isomorphism $H_2(\pi_{g,0}, \mathbb{Z}) \cong H_2(\Sigma_{g,0}, \mathbb{Z})$ of Theorem B.8. Integration against $[\pi_{g,0}]$ gives an isomorphism

$$[\pi_{g,0}] \frown : H^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}.$$

Let $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ be given by the composition of the quotient map $Z^2(\pi_{g,0}, \mathbb{R}) \to H^2(\pi_{g,0}, \mathbb{R})$ and the integration against $[\pi_{g,0}]$. Clearly, φ vanishes on $B^2(\pi_{g,0}, \mathbb{R})$.

Lemma 4.1.1. *Let $\Gamma = \pi_{g,0}$ be a closed surface group. The composition of $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ with the form ω of (4.0.1) defines a 2-form on $\operatorname{Hom}(\pi_{g,0}, G)$ whose kernel is $B^1(\pi_{g,0}, \mathbb{R})$.*

Proof. The proof relies on Poincaré duality in group cohomology for the group $\pi_{g,0}$. It implies that the cup product

$$H^1(\pi_{g,0}, \mathbb{R}) \times H^1(\pi_{g,0}, \mathbb{R}) \xrightarrow{\smile} H^2(\pi_{g,0}, \mathbb{R})$$

is a nondegenerate pairing. This means that the form $\varphi \circ \omega$ is degenerate on $B^1(\pi_{g,0}, \mathbb{R})$ only. $\square$

The induced nondegenerate closed form $(\omega_{\mathcal{G}})_\phi \colon H^1(\pi_{g,0}, \mathfrak{g}_\phi) \times H^1(\pi_{g,0}, \mathfrak{g}_\phi) \to \mathbb{R}$ is the celebrated Goldman symplectic form for character varieties of closed surface groups representations. The original argument of Goldman in [Gol84] to prove that the $\omega_{\mathcal{G}}$ is closed is inspired by the treatment of the case when G is compact in [AB83]. The proof involves an infinite dimensional symplectic reduction from the affine space of connections on some vector bundle, see [Gol84] and [Lab13, §6] for more details.

Remark 4.1.2. The Goldman symplectic form depends on the pairing B on the Lie algebra of G. Different choices of pairing for the same Lie group G may lead to different symplectic structures. Abusing once again of the term "symplectic manifold", one can say that Goldman's construction is a functor form the product category of the category of closed connected oriented surfaces $\Sigma_{g,0}$ with the category of quadrable Lie groups G with a choice of a form pairing B to the category of "symplectic manifold"

$$\big(\Sigma_{g,0}, (G, B)\big) \rightsquigarrow \big(\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G), \omega_{\mathcal{G}}\big).$$

We point out that the quotients $\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G)$ obtained for different choices of basepoints in $\Sigma_{g,0}$ are naturally isomorphic (the isomorphism does *not* depend on the choice of path connecting different basepoints).

4.2 General surface groups

Let $\Gamma = \pi_{g,n}$ be a surface group. We will assume in this section that $n > 0$. As mentioned earlier, in that case $\pi_{g,n}$ is a free group and the representation variety $\mathrm{Hom}(\pi_{g,n}, G)$ is isomorphic to the product G^{2g+n-1}. It can be written as the disjoint union of so-called relative representation varieties.

Definition 4.2.1 (Relative representation variety)**.** Let $\mathcal{C} = (C_1, \ldots, C_n)$ be an ordered collection of n conjugacy classes in G. The *relative representation variety* associated to $(\pi_{g,n}, G, \mathcal{C})$ is the subspace of $\mathrm{Hom}(\pi_{g,n}, G)$ given by

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G) := \{\phi \in \mathrm{Hom}(\pi_{g,n}, G) : \phi(c_i) \in C_i,\ \forall i = 1, \ldots, n\},$$

where $c_1, \ldots, c_n$ refer to the generators of $\pi_{g,n}$ in the presentation (1.1.3).

If G/G denotes the set of conjugacy classes in G, then

$$\mathrm{Hom}(\pi_{g,n}, G) = \bigsqcup_{\mathcal{C} \in (G/G)^n} \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G).$$

Relative character varieties are really associated to the particular presentation of $\pi_{g,n}$ that we fixed in (1.1.3). The conjugation action of G on $\mathrm{Hom}(\pi_{g,n}, G)$ restricts to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$.

Lemma 4.2.2. *Let G be a Lie group equipped with an analytic atlas. The relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally an analytic subvariety of G^{2g+n}. If G is a complex algebraic group, then $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an algebraic subvariety of $\mathrm{Hom}(\pi_{g,n}, G)$. If G is a real algebraic group, then $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is a semialgebraic subvariety of $\mathrm{Hom}(\pi_{g,n}, G)$.*

Proof. The proof is analogous to the proof of Lemma 1.2.2. A conjugacy class $C \in G/G$ is a smooth submanifold of G isomorphic to $G/Z(c)$, where c is any element of C (recall that $Z(c)$ is a closed subgroup of G). It has a unique structure of real analytic manifold that makes the projection map $G \to G/Z(c)$ an analytic submersion. The relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally identified with the subspace of $G^{2g} \times C_1 \times \ldots \times C_n$ cut out by the single relation of the surface group $\pi_{g,n}$ (see (1.1.3)). This shows that $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an analytic subvariety of G^{2g+n}. Observe now that, if G is a complex algebraic group, then conjugacy classes in G are algebraic subvarieties of G. This can be seen as a consequence of Chevalley's Theorem. Moreover, if G is a real algebraic group, then conjugacy classes in G are semialgebraic subvarieties of G[1]. This, in turn, is a consequence of Tarski–Seidenberg Theorem. □

We would like to determine the Zariski tangent space to relative character varieties. We follow the approach of [GHJW97, §4]. Let $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. The Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ at ϕ is the space of all tangent vectors in $Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to a smooth deformation ϕ_t of ϕ inside $\mathrm{Hom}(\pi_{g,n}, G)$ that satisfies $\phi_t(c_i) \in C_i$ up to first order. Observe that the condition $\phi_t(c_i) \in C_i$ is equivalent to the existence of a smooth 1-parameter family $g_i(t) \in G$, with $g_i(0) = 1$, and

$$\phi_t(c_i) = g_i(t)\phi(c_i)g_i(t)^{-1}. \tag{4.2.1}$$

[1]An example of conjugacy classes that are a semialgebraic subvarieties, but not algebraic subvarieties, are parabolic conjugacy classes inside $\mathrm{SL}(2, \mathbb{R})$.

Lemma 4.2.3. *A vector $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to ϕ_t at $t = 0$ satisfies (4.2.1) up to first order if and only if*

$$v(c_i) = \dot{g}_i - \mathrm{Ad}(\phi(c_i))\dot{g}_i,$$

where $\dot{g}_i \in \mathfrak{g}$ is the tangent vector to $g_i(t)$ at $t = 0$.

Proof. We use $\frac{d}{dt}\big|_{t=0} \phi_t(c_i)\phi(c_i)^{-1} = v(c_i)$ and derive the relation (4.2.1). □

Corollary 4.2.4 ([GHJW97])**.** *The Zariski tangent space to* $\mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ *at ϕ is*

$$T_\phi \mathrm{Hom}_{\mathcal{C}}(\Gamma, G) = \left\{ v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi) \,:\, \forall i = 1, \ldots, n,\ \exists \xi_i \in \mathfrak{g},\ v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i \right\}.$$

The cocycles $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ that satisfy the property stated in the conclusion of Corollary 4.2.4 are called *parabolic 1-cocycles*, see Appendix B.8. The subspace of parabolic cocycles is denoted

$$Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \subset Z^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

The tangent space to the G-orbit $\mathcal{O}_\phi$ of $\phi \in \mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ still identifies with $B^1(\pi_{g,n}, \mathfrak{g}_\phi)$. The quotient of parabolic 1-cocycles by 1-coboundaries is the first parabolic group cohomology group of $\pi_{g,n}$ with coefficients in the $\pi_{g,n}$-module $\mathfrak{g}_\phi$:

$$H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) = Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi)/B^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

Proposition 4.2.5. *Let G be a quadrable Lie group. The dimension of the Zariski tangent space to* $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ *at ϕ is*

$$(2g - 1)\dim G + \sum_{i=1}^{n} \dim C_i + \dim Z(\phi).$$

In particular, the smooth points of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ *are the representations ϕ such that*

$$\dim Z(G) = \dim Z(\phi).$$

Proof. We proceed as in the alternative proof of Proposition 1.5.3. Let $A_i = \phi(a_i)$, $B_i = \phi(b_i)$ and $R_i = \phi(c_i)$, where a_i, b_i, c_i refer to the presentation (1.1.3). Consider the map $\mu\colon \mathfrak{g}^{2g+n} \to \mathfrak{g}$ obtained by differentiating the unique surface group relation:

$$\begin{aligned}
\mu(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n) = &\sum_{i=1}^{g} \left(\prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\alpha_i - \mathrm{Ad}(A_i B_i A_i^{-1})\alpha_i) \\
&- \sum_{i=1}^{g} \left(\prod_{j\leqslant i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\beta_i - \mathrm{Ad}(B_i A_i B_i^{-1})\beta_i) \\
&+ \prod_{k=1}^{g} \mathrm{Ad}\left([A_k, B_k]\right) \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \mathrm{Ad}\left(R_j\right) \right) (\gamma_i - \mathrm{Ad}(R_i)\gamma_i).
\end{aligned}$$

Let V be the orthogonal complement of the image of μ with respect to the pairing B. Similarly as in the alternative proof of Proposition 1.5.3, we conclude that $V = \mathfrak{z}(\phi)$. The Rank-Nullity

Theorem gives

$$\dim T_\phi \operatorname{Hom}_{\mathcal{C}}(\Gamma, G) = \dim \operatorname{Ker}(\mu) = (2g-1)\dim G + \sum_{i=1}^n \dim C_i + \dim Z(\phi). \qquad \square$$

Remark 4.2.6. We make a little digression on the dimension of conjugacy orbits inside Lie groups. Recall that any conjugacy class $\mathcal{C} \in G/G$ is a smooth submanifold of G diffeomorphic to the quotient $G/Z(g)$. If G is quadrable, the pairing B on $\mathfrak{g}$ can be used to identify coadjoint orbits in $\mathfrak{g}^*$ to adjoint orbits in $\mathfrak{g}$. Coadjoint orbits are naturally symplectic, see e.g. [CdS01, Homework 17]. The exponential map maps the adjoint orbit of $\xi \in \mathfrak{g}$ to the conjugacy orbit of $\exp(\xi)$ in G. Recall however that the Lie theoretic exponential map needs not be a local diffeomorphism at ξ. If it were, it would imply that the conjugacy orbit of $\exp(\xi)$ in G has even dimension. M. Riestenberg pointed out to the author a class of examples of Lie groups that contain conjugacy classes of odd dimension. They consist of the group of all isometries of an odd-dimensional symmetric space X. In that case, the conjugacy class of the orientation-reversing isometry s_p that reflects through a point p is the set of all the orientation-reversing isometries s_q for $q \in X$ and is therefore isomorphic to X.

Question 4.2.7. When does a conjugacy orbit in a quadrable Lie group G have even dimension? Is it necessarily the case if it lies in the image of the exponential map?

We close the digression and go back to relative representation varieties. We would like to obtain an analogue of the Goldman symplectic form for general surface groups. We denote by $\partial_i \pi_{g,n}$ the subgroup of $\pi_{g,n}$ generated by c_i. We write $\partial \pi_{g,n}$ for the collection of subgroups $\{\partial_i \pi_{g,n}\}$. Observe that the cup product in group cohomology restricts to the product (B.15) in parabolic group cohomology. It gives an anti-symmetric bilinear form

$$\omega\colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \xrightarrow{\smile} Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi) \xrightarrow{B_*} Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}).$$

Let $[\pi_{g,n}]$ be a generator of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}$, that corresponds to a choice of orientation for the surface $\Sigma_{g,n}$. Integrating against the fundamental class $[\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$. Let $\varphi\colon Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to \mathbb{R}$ be the composition of the quotient map $Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R})$ with the integration against $[\pi_{g,n}]$. Similarly as in the closed case, it was proven in [GHJW97, §3] that the 2-form $\varphi \circ \omega$ is degenerate precisely on $B^1(\pi_{g,n}, \mathfrak{g}_\phi)$ and is furthermore closed [GHJW97, Thm. 7.1] (see also [Law09]). We obtain

Theorem 4.2.8 ([GHJW97])**.** *Let $\Gamma = \pi_{g,n}$ be a surface group. The composition of*

$$\omega\colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R})$$

with $\varphi\colon Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to \mathbb{R}$ gives a nondegenerate closed 2-form

$$(\omega_{\mathcal{G}})_\phi\colon H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to \mathbb{R}$$

Definition 4.2.9 (Relative character varieties)**.** The Hausdorffization of the topological quotient

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\,\mathrm{Inn}(G)$$

is called the *relative character variety* associated to $(\pi_{g,n}, G, \mathcal{C})$. The nondegenerate closed 2-form $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\,\mathrm{Inn}(G)$.

Depending on the properties of the group G, the definition of relative character variety can be refined in order to get a better control of its structure similarly as in Section 3.

Remark 4.2.10 (Poisson structure). The representation variety $\mathrm{Hom}(\pi_{g,n}, G)$ is the disjoint union of all the relative representation varieties $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ over all possible choices for $\mathcal{C} \in (G/G)^n$. The quotient of each relative representation variety by the $\mathrm{Inn}(G)$-action has a symplectic structure in the sense of Theorem 4.2.8. It turns out that these quotients are the symplectic leaves of a Poisson structure on the quotient of the representation variety by the $\mathrm{Inn}(G)$-action. The reader is referred to [BJ21] for a precise statement, a proof, and references to prior proofs.

Definition 4.2.11 (Goldman symplectic measure)**.** Both in the case of character varieties for closed surfaces and in the case of relative character varieties for punctured surfaces, the measure obtained from the Goldman symplectic form is denoted $\nu_{\mathcal{G}}$ and called the *Goldman symplectic measure.*

The Goldman symplectic measure is a strictly positive Borel measure. It means that open sets are measurable and always have positive measure if they are nonempty.

4.2.1 Case of a punctured sphere

In the case that $\Gamma = \pi_{0,n}$ is the fundamental group of a punctured sphere, then one can obtain fairly explicit formulae for the Goldman symplectic form on $\mathrm{Hom}_{\mathcal{C}}(\pi_{0,n}, G)$. We abbreviate $\pi_n := \pi_{0,n}$ in this section. We first need to compute a fundamental class $[\pi_n]$ explicitly. All computations are lead in the bar complex for group cohomology introduced in Appendix B.2.

Lemma 4.2.12. *Let $e \in \mathbb{Z}[\pi_n \times \pi_n]$ be given by*

$$e := (c_1, c_2) + (c_1c_2, c_3) + \ldots + (c_1c_2 \cdot \ldots \cdot c_{n-1}, c_n) + (1, 1). \tag{4.2.2}$$

Then $(e, c_1, \ldots, c_n) \in Z^2(\pi_n, \partial\pi_n, \mathbb{Z})$, i.e. the 2-chain $(e, c_1, \ldots, c_n)$ is closed. Moreover, $[(e, c_1, \ldots, c_n)]$ is a generator of $H_2(\pi_n, \partial\pi_n, \mathbb{Z})$.

Proof. Let $\imath_i \colon \partial_i\pi_n \hookrightarrow \pi_n$ denote the inclusion of the subgroup $\partial_i\pi_n$ (generated by c_i) into π_n. The long exact sequence (B.9) in group homology for the pair $(\pi_n, \partial\pi_n)$ contains

$$\ldots \to H_2(\pi_n, \mathbb{Z}) \longrightarrow H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \xrightarrow{\delta} H_1(\partial\pi_n, \mathbb{Z}) \xrightarrow{\oplus\imath_i} H_1(\pi_n, \mathbb{Z}) \to \ldots.$$

Since $H_2(\pi_n, \mathbb{Z}) = 0$, the connecting morphism δ is an isomorphism onto its image. Hence $H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \cong \mathrm{Ker}(\oplus\imath_i)$. Recall that $H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \cong \mathbb{Z}$, and so $\mathrm{Ker}(\oplus\imath_i) \cong \mathbb{Z}$. The strategy to find a fundamental class is to first find an isomorphism $\psi \colon \mathrm{Ker}(\oplus\imath_i) \to \mathbb{Z}$, then compute $\psi^{-1}(1) \in H_1(\partial\pi_n, \mathbb{Z})$ and finally compute its preimage under δ.

Recall that the bar chain complex that computes the homology of the group π_n with coefficients in the trivial π_n-module $\mathbb{Z}$ is defined by $C_k(\pi_n, \mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_n^k] \cong \mathbb{Z}[\pi_n^k]$, where $\pi_n^k = \pi_n \times \ldots \times \pi_n$. The differentials in degrees 1 and 2 are

$$\begin{array}{ccccc} C_2(\pi_n\mathbb{Z}) & \xrightarrow{\partial} & C_1(\pi_n, \mathbb{Z}) & \xrightarrow{\partial} & C_0(\pi_n, \mathbb{Z}) \\ & & g & \longmapsto & 0 \\ (g,h) & \longmapsto & g + h - gh. & & \end{array}$$

In particular, the first homology group is

$$H_1(\pi_n, \mathbb{Z}) = \mathbb{Z}[\pi_n]/(g + h - gh), \tag{4.2.3}$$

Since $c_n = \prod_{i=1}^{n-1} c_i^{-1}$ by construction, it holds that $c_n = \sum_{i=1}^{n-1} -c_i$ and $c_i^k = k \cdot c_i$ inside $\mathbb{Z}[\pi_n]/(g + h - gh)$. This gives an isomorphism $\mathbb{Z}[\pi_n]/(g+h-gh) \cong \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_{n-1}$. For the same reason,

$$H_1(\partial_i \pi_n, \mathbb{Z}) = \mathbb{Z}[\partial_i \pi_n]/(g + h - gh) \cong \mathbb{Z} \cdot c_i.$$

We are interested in the morphism $\varphi \colon \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ induced by $\oplus \iota_i$ in the following diagram

$$\begin{tikzcd} H_1(\partial\pi_n, \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\oplus\iota_i"] & \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_n \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}^n \arrow[d, dashed, "\varphi"] \\ H_1(\pi_n, \mathbb{Z}) \arrow[r, "\cong"] & \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_{n-1} \arrow[r, "\cong"] & \mathbb{Z}^{n-1} \end{tikzcd}$$

The previous identifications implies that φ is the morphism

$$\varphi(m_1, \ldots, m_n) = (m_1 - m_n, \ldots, m_{n-1} - m_n).$$

Therefore, the kernel of φ consists of vectors having identical entries and thus $\mathrm{Ker}(\oplus\iota_i)$ is generated by $[(c_1, \ldots, c_n)] \in H_1(\partial\pi_n, \mathbb{Z})$.

It remains to compute $\delta^{-1}([(c_1, \ldots, c_n)])$. Since δ is induced from the projection $\mathbb{Z}[\pi_n^2] \oplus \mathbb{Z}[\partial\pi_n] \to \mathbb{Z}[\partial\pi_n]$, it is enough to find a chain $e \in \mathbb{Z}[\pi_n^2]$ such that $(e, c_1, \ldots, c_n)$ is closed. This is the case for e given by (4.2.2) because $\partial_2 e = -c_1 - \ldots - c_n$ and hence $\partial_2(e, c_1, \ldots, c_n) = 0$. □

The fundamental class $[\pi_n]$ was already computed in [GHJW97, Section 2] using different methods. We now give explicit formulae for the Goldman symplectic form.

Let $u, v \in Z^1_{par}(\pi_n, \mathfrak{g}_\phi)$. By definition of parabolic cocycles, there exist $\xi_i, \zeta_i \in \mathfrak{g}$ such that

$$u(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i, \quad v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i, \quad i = 1, \ldots, n.$$

The first step consists in computing a preimage of u inside $Z^1(\pi_n, \partial\pi_n, \mathfrak{g}_\phi)$. Note that

$$\partial\xi_i(c_i) = \mathrm{Ad}(\phi(c_i))\xi_i - \xi_i = -u(c_i).$$

Hence, the 1-cochain $(u, -\xi_1, \ldots, -\xi_n)$ is closed and is a preimage of u.

To compute $\omega_{\mathcal{G}}(u, v)$, we proceed as follows:

1. Apply the cup product to $(u, -\xi_1, \ldots, -\xi_n)$ and v.

2. Apply the pairing B-

3. Take the cap product with the fundamental form $[\pi_n]$ computed in Lemma 4.2.12 (here we use Lemma B.15).

This gives

$$\omega_G(u,v) = B_*(u \smile v)(e) + \sum_{i=1}^{n} B_*(\xi_i \smile v)(c_i). \tag{4.2.4}$$

We develop each cup product according to (B.11) and plug in the value of e computed in Lemma 4.2.12. The right-hand side of (4.2.4) becomes

$$\sum_{i=2}^{n} B(u(c_1 \cdot \ldots \cdot c_{i-1}) \cdot \mathrm{Ad}(\phi(c_1 \cdot \ldots \cdot c_{i-1}))v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{4.2.5}$$

We can further simplify (4.2.5) using to the Ad-invariance of B and the formula $u(x^{-1}) = -\mathrm{Ad}(\phi(x^{-1}))u(x)$. It is useful to introduce the notation $b_{i-2} := c_{i-1}^{-1} \cdots c_1^{-1}$. In particular, $b_0 = c_1^{-1}$ and $b_{n-1} = 1$. We obtain

$$\omega_{\mathcal{G}}(u,v) = -\sum_{i=2}^{n} B(u(b_{i-2}) \cdot v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{4.2.6}$$

Using that ω_G and the cup product are anti-symmetric, we get the following equivalent form of (4.2.6)

$$\omega_{\mathcal{G}}(u,v) = -\sum_{i=2}^{n} B(u(b_{i-2}) \cdot v(c_i)) - \sum_{i=1}^{n} B(\zeta_i \cdot u(c_i)). \tag{4.2.7}$$

Formulae (4.2.4), (4.2.7), and (4.2.6), were already obtained in the proof of [GHJW97, Key Lemma 8.4]. We go one step further.

Lemma 4.2.13. *It holds that*

$$\omega_{\mathcal{G}}(u,v) = \sum_{i=1}^{n-2} B((\zeta_{i+1} - \zeta_{i+2}) \cdot u(b_i)). \tag{4.2.8}$$

Proof. Using $v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i$ and the Ad-invariance of B, we get

$$B(u(b_{i-2}) \cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2}) \cdot \zeta_i)$$

By construction, $b_{i-1} = c_i^{-1} b_{i-2}$ and thus $u(b_{i-1}) = u(c_i^{-1}) + \mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2})$. So,

$$B(u(b_{i-2}) \cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\zeta_i \cdot u(b_{i-1})) + B(\zeta_i \cdot u(c_i^{-1})).$$

Therefore, (4.2.7) becomes

$$\begin{aligned}\omega_{\mathcal{G}}(u,v) =& \sum_{i=2}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\ & - B(\zeta_1 \cdot u(c_1)) - \sum_{i=2}^{n} B(\zeta_i \cdot (u(c_i^{-1}) + u(c_i)) \\ =& B(\zeta_2 \cdot u(b_1)) + \sum_{i=3}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\ & - \sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big) \\ =& \sum_{i=1}^{n-2} B\big((\zeta_{i+1} - \zeta_{i+2}) \cdot u(b_i)\big) - \underbrace{\sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big)}_{=:\Omega},\end{aligned}$$

where in the second equality we used $b_0 = c_1^{-1}$ and in the third equality that $u(b_{n-1}) = u(1) = 0$. It remains to prove that $\Omega = 0$. Using $u(x^{-1}) = -\operatorname{Ad}(\phi(x^{-1}))u(x)$, we get

$$B(\zeta_i \cdot u(c_i^{-1})) = -B(\operatorname{Ad}(\phi(c_i))\zeta_i \cdot u(c_i)).$$

Therefore, using $v(c_i) = \zeta_i - \operatorname{Ad}(\phi(c_i))\zeta_i$, we conclude

$$\Omega = \sum_{i=1}^{n} B(u(c_i) \cdot v(c_i)).$$

By construction, $B(u(\cdot) \cdot v(\cdot))$ defines a 1-cocycle in $Z^1(\pi_n, \mathbb{R})$. Closeness can also be computed directly using (B.2), similarly as in the proof of Lemma B.11. Therefore, Ω is equal to the evaluation of the 1-cocycle $B(u(\cdot) \cdot v(\cdot))$ on the 1-cycle $c_1 + \ldots + c_n$. The identification (4.2.3) shows that the 1-cycle $\sum_{i=1}^{n} c_i$ vanishes in homology (this is a consequence of the fact that $\prod_{i=1}^{n} c_i = 1$). Hence, $\Omega = B(u(1) \cdot v(1)) = 0$ as desired. $\square$

Chapter 5

Volume of a representation

The topology of a representation variety is notably known to be complicated. The enumeration of its connected components is a non-trivial task. The volume of a representation is an invariant that lets us approach this problem. We recall its definition below and recommend [BIW10] for more details.

5.1 Definition

The volume is defined in [BIW10] for representations of surface groups $\Gamma = \pi_{g,n}$ into Hermitian Lie groups G. Recall that a *Hermitian Lie group* G is a semisimple Lie group, with finite center and no compact factors, such that its associated symmetric space X is a Hermitian manifold. The Kähler form obtained from the unique G-invariant Hermitian metric of constant sectional curvature -1 on X is denoted ω_X. The classical examples of Hermitian Lie groups include $\mathrm{SU}(p,q)$ and $\mathrm{Sp}(2n,\mathbb{R})$.

Example 5.1.1. The guiding example in this section is the group $G = \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. It is a simple Lie group, without compact factor and with center $Z(\mathrm{SL}(2,\mathbb{R})) = \{\pm I\}$. It is of Hermitian type. It is sometimes more convenient to consider the center-free quotient $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ instead, which is also of Hermitian type. The associated symmetric space is the upper half-plane $X = \mathbb{H}$ on which $\mathrm{SL}(2,\mathbb{R})$ acts by Möbius transformations, see Appendix A for more considerations on the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$. The group of orientation-preserving isometries of $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$. The associated Kähler form is $\omega_{\mathbb{H}} = (dx \wedge dy)/y^2$.

Let G be a Hermitian Lie group with symmetric space X. Given three points z_1, z_2, z_3 in X, we denote by $\Delta(z_1, z_2, z_3)$ the oriented geodesic triangle in X with vertices z_1, z_2, z_3. Its signed area, computed with the area form associated to ω_X, is denoted by

$$[\Delta(z_1, z_2, z_3)] := \int_{\Delta(z_1,z_2,z_3)} \omega_X.$$

Fix a basepoint $z \in X$ and consider the function

$$\begin{aligned} c\colon G \times G &\to \mathbb{R} \\ (g_1, g_2) &\to \big[\Delta\big(z, g_1 z, g_1 g_2 z\big)\big]. \end{aligned} \tag{5.1.1}$$

Lemma 5.1.2. *The function c satisfies the cocycle condition*

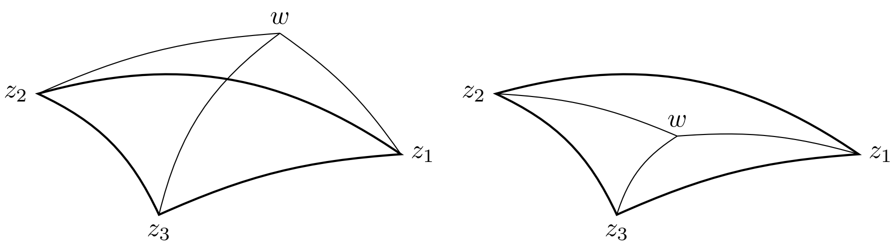
$$c(g_2, g_3) - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2) = 0 \tag{5.1.2}$$

for every $g_1, g_2, g_3 \in G$, compare (B.2).

Proof. We need the following identity: if z_1, z_2, z_3 are any three points in X, then, for any fourth point $w \in X$,

$$[\Delta(z_1, z_2, z_3)] = [\Delta(z_1, z_2, w)] + [\Delta(z_2, z_3, w)] + [\Delta(z_3, z_1, w)]. \tag{5.1.3}$$

The following picture should convince the reader of (5.1.3).



In terms of triangle areas, the cocycle condition (5.1.2) is equivalent to

$$[\Delta(z, g_2 z, g_2 g_3 z)] + [\Delta(z, g_1 z, g_1 g_2 g_3 z)]$$

being equal to

$$[\Delta(z, g_1 g_2 z, g_1 g_2 g_3 z)] + [\Delta(z, g_1 z, g_1 g_2 z)].$$

Since $g_1 \in G$ acts by isometry on X and preserves the orientation, the latter is equivalent to

$$[\Delta(g_1 z, g_1 g_2 z, g_1 g_2 g_3 z)] + [\Delta(z, g_1 z, g_1 g_2 g_3 z)]$$

being equal to

$$[\Delta(z, g_1 g_2 z, g_1 g_2 g_3 z)] + [\Delta(z, g_1 z, g_1 g_2 z)].$$

This is precisely formula (5.1.3) applied to $z_1 = z$, $z_2 = g_1 z$, $z_3 = g_1 g_2 z$ and $w = g_1 g_2 g_3 z$. $\square$

Lemma 5.1.2 implies that c defines a cohomology class $\kappa := [c]$ inside $H^2(G, \mathbb{R})$. The function c is bounded because the area of a geodesic triangle in X is bounded. This means that the cohomology class κ gives a class $\kappa \in H^2_b(G, \mathbb{R})$ in the second bounded cohomology group of G. We recommend [Löh10] for an introduction to bounded group cohomology.

Lemma 5.1.3. *The cohomology class κ is independent of the choice of the basepoint z involved in the definition of c (whereas c does depend on the point z).*

Proof. For the purpose of this proof, we will write c_z instead of c for the cocycle (5.1.1) to emphasize the dependence on the basepoint z. Given another basepoint $x \in X$, we prove that $c_z - c_x$ is a coboundary.

First, we develop $c_z(g_1, g_2) = [\Delta(z, g_1 z, g_1 g_2 z)]$ using (5.1.3) with $w = g_1 x$. We obtain

$$\begin{aligned} c_z(g_1, g_2) &= [\Delta(z, g_1 z, g_1 x)] + [\Delta(g_1 z, g_1 g_2 z, g_1 x)] + [\Delta(g_1 g_2 z, z, g_1 x)] \\ &= -[\Delta(x, z, g_1^{-1} z)] + [\Delta(x, z, g_2 z)] + [\Delta(g_1 g_2 z, z, g_1 x)]. \end{aligned}$$

Now, we develop $[\Delta(g_1 g_2 z, z, g_1 x)]$ using (5.1.3) with $w = x$. This gives

$$\begin{aligned} [\Delta(g_1 g_2 z, z, g_1 x)] &= [\Delta(g_1 g_2 z, z, x)] + [\Delta(z, g_1 x, x)] + [\Delta(g_1 x, g_1 g_2 z, x)] \\ &= -[\Delta(x, z, g_1 g_2 z)] - [\Delta(z, x, g_1 x)] + [\Delta(g_1 x, g_1 g_2 z, x)]. \end{aligned}$$

Finally, we develop $[\Delta(g_1 x, g_1 g_2 z, x)]$ using (5.1.3) with $w = g_1 g_2 x$. We have

$$\begin{aligned} [\Delta(g_1 x, g_1 g_2 z, x)] &= [\Delta(g_1 x, g_1 g_2 z, g_1 g_2 x)] + [\Delta(g_1 g_2 z, x, g_1 g_2 x)] + [\Delta(x, g_1 x, g_1 g_2 x)] \\ &= [\Delta(z, x, g_2^{-1} x)] - [\Delta(z, x, g_2^{-1} g_1^{-1} x)] + c_x(g_1, g_2). \end{aligned}$$

Consider the 1-cochain $v_{x,z}(g) := [\Delta(x, z, gz)]$. It holds that

$$\partial v_{x,z}(g_1, g_2) = [\Delta(x, z, g_1 z)] + [\Delta(x, z, g_2 z)] - [\Delta(x, z, g_1 g_2 z)].$$

In particular, $\partial v_{x,z}(g, g^{-1}) = [\Delta(x, z, gz)] + [\Delta(x, z, g^{-1} z)]$. The previous computations show that

$$c_z(g_1, g_2) - c_x(g_1, g_2) = \partial v_{x,z}(g_1, g_2) - \partial v_{x,z}(g_1, g_1^{-1}) + \partial v_{z,x}(g_2^{-1}, g_1^{-1}) - \partial v_{z,x}(g_1, g_1^{-1}).$$

We conclude as predicted that $c_z - c_x$ is a coboundary. $\square$

Given a representation $\phi\colon \pi_{g,n} \to G$, we can pull back κ to the class $\phi^*\kappa$ inside $H_b^2(\pi_{g,n}, \mathbb{R})$. An important property of the bounded cohomology of the group $\pi_{g,n}$ is that the map

$$j\colon H_b^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to H_b^2(\pi_{g,n}, \mathbb{R}) \tag{5.1.4}$$

from the long exact sequence in cohomology for the pair $(\pi_{g,n}, \partial\pi_{g,n})$ is an isomorphism, see [Löh10, Thm. 2.6.14]. Recall finally that integrating along a fundamental class $[\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \cong \mathbb{R}$.

Definition 5.1.4 (Volume of a representation, [BIW10])**.** Let G be a Hermitian Lie group. The *volume* of a representation[1] $\phi\colon \pi_{g,n} \to G$ is the real number defined by

$$\mathrm{vol}(\phi) := j^{-1}(\phi^*\kappa) \frown [\pi_{g,n}].$$

The volume is a generalization of the Euler number of a representation of a closed surface group

[1]Up to a constant, the volume of a representation ϕ is sometimes called the *Toledo number* of the representation and is, in that case, denoted $\mathrm{Tol}(\phi)$. The two notions are related by the identity $\mathrm{vol}(\phi) = 2\pi\,\mathrm{Tol}(\phi)$.

into $\mathrm{PSL}(2,\mathbb{R})$. The latter is equal to the Euler number of the flat $\mathbb{RP}^1$-bundle $(\widetilde{\Sigma}_{g,0} \times \mathbb{RP}^1)/\pi_{g,0} \to \Sigma_{g,0}$ associated to a representation $\pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$.

5.2 Properties

Lemma 5.2.1. *The volume is invariant under the conjugation action of G on $\mathrm{Hom}(\pi_{g,n}, G)$ and thus descends to a function*

$$\mathrm{vol}\colon \mathrm{Hom}(\pi_{g,n}, G)/\,\mathrm{Inn}(G) \to \mathbb{R}.$$

Proof. Consider the cocycle c defined in (5.1.1). The diagonal conjugation action of an element $g \in G$ on $G \times G$ amounts to a change of basepoint in the definition of c. Indeed, if c_z denotes the cocycle (5.1.1) defined using the basepoint $z \in X$, then it holds that $c_z(gg_1g^{-1}, gg_2g^{-1}) = c_{g^{-1}z}(g_1, g_2)$. Since, by Lemma 5.1.3, the cohomology class κ is independent of the choice of the basepoint defining c, we conclude that the volume is an invariant of conjugation. ☐

The main properties of the volume are the following. We denote by $\chi(\Sigma_{g,n})$ the Euler characteristic of $\Sigma_{g,n}$.

Theorem 5.2.2 ([BIW10]). *The volume, seen as a function* $\mathrm{vol}\colon \mathrm{Hom}(\pi_{g,n}, G) \to \mathbb{R}$*, has the following properties:*

1. vol *is a continuous function.*
2. vol *is locally constant on each relative representation variety.*
3. (Milnor–Wood inequality) vol *is bounded:*
$$|\,\mathrm{vol}\,| \leqslant 2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \mathrm{rank}(G),$$
moreover, if $n > 0$, then vol *is a surjective function onto the interval*
$$\left[-2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \mathrm{rank}(G),\, 2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \mathrm{rank}(G)\right].$$
4. vol *is additive: if $\Sigma_{g,n}$ is separated by a simple closed curve into two surfaces S_1 and S_2, then, for every $\phi \in \mathrm{Hom}(\pi_{g,n}, G)$,*
$$\mathrm{vol}(\phi) = \mathrm{vol}(\phi\restriction_{\pi_1(S_1)}) + \mathrm{vol}(\phi\restriction_{\pi_1(S_2)}).$$

The first and second statement in Theorem 5.2.2 imply that the set of representations of a given volume forms a collection of connected components of each relative character variety. Recall that in the case of a closed surface group and $G = \mathrm{PSL}(2,\mathbb{R})$, the Euler number completely distinguishes the connected components of the character variety [Gol88].

The volume has an interesting symmetry that comes from reversing the orientation of X. By definition, for each $z \in X$, there exists an orientation-reversing isometry s_z of X that fixes z. This gives an involutive automorphism $\sigma\colon G \to G$ defined by $\sigma(g) := s_z \circ g \circ s_z$. Indeed, if $g \in G$ is an orientation-preserving isometry of X, then $s_z \circ g \circ s_z$ is again an orientation-preserving isometry of

X, and hence belongs to G. Using the functoriality of representation varieties (see Lemma 1.2.7), the involution σ descends to an analytic involution

$$\sigma\colon \operatorname{Hom}(\pi_{g,n}, G) \to \operatorname{Hom}(\pi_{g,n}, G).$$

Lemma 5.2.3. *The involution σ satisfies the following properties:*

1. *σ preserves conjugacy classes of representations, and therefore descends to an involution*

$$\overline{\sigma}\colon \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G) \to \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G).$$

2. *σ depends on the choice of $z \in X$ only up to conjugation, in particular, $\overline{\sigma}$ is independent of the choice of $z \in X$.*

3. *For any representation $\phi \in \operatorname{Hom}(\pi_{g,n}, G)$ it holds that*

$$\operatorname{vol}(\sigma(\phi)) = -\operatorname{vol}(\phi).$$

Proof. The first assertion follows from $\sigma(g\phi g^{-1}) = (s_z \circ g \circ s_z)\sigma(\phi)(s_z \circ g^{-1} \circ s_z)$ and the observation that $s_z \circ g \circ s_z$ is orientation-preserving. If $z' \in X$ is a second point, then it holds that $s_{z'} \circ g \circ s_{z'} = (s_{z'} \circ s_z)(s_z \circ g \circ s_z)(s_z \circ s_{z'})$, which proves the second assertion because $s_{z'} \circ s_z$ is orientation-preserving. Finally, note that $(\sigma(\phi))^*\kappa = \phi^*(\sigma^*\kappa)$ and $\sigma^*\kappa = -\kappa$ because s_z reverses the orientation of X. $\square$

Example 5.2.4. Consider the case $G = \mathrm{SL}(2, \mathbb{R})$. An example of orientation-reversing isometry of the upper half-plane is given by $z \mapsto -\overline{z}$. It fixes the imaginary axis. The associated involutive automorphism σ of $\mathrm{SL}(2, \mathbb{R})$ is given by conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of determinant -1.

The involution $\sigma\colon \operatorname{Hom}(\pi_{g,n}, G) \to \operatorname{Hom}(\pi_{g,n}, G)$ maps the relative representation variety $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ to the relative representation variety $\operatorname{Hom}_{\sigma(\mathcal{C})}(\pi_{g,n}, G)$. Since G is of Hermitian type, it is by definition semisimple and hence quadrable. The Goldman symplectic form built from the Killing form on $\mathfrak{g}$ is invariant under σ. This is a consequence of the fact that the Killing form is invariant under automorphisms of $\mathfrak{g}$. In this case, the involution $\sigma\colon G \to G$ induces an automorphism $D\sigma\colon \mathfrak{g} \to \mathfrak{g}$.

5.3 Alternative definition

A downside of Definition 5.1.4 is the lack of computability. Given a representation $\phi\colon \pi_{g,n} \to G$, computing $j^{-1}(\phi^*\kappa)$ means finding a primitive in $H^1(\partial_i \pi_{g,n}, \mathbb{R})$ for each restriction $\phi^*\kappa\restriction_{\partial_i \pi_{g,n}}$. This is a non-trivial task in general. There is an alternative definition of the volume of a representation that makes it easier to compute. It is based on a notion of rotation number that generalizes the classical notion of rotation number for homeomorphisms of the circle, see [Ghy01] for an exposition of the classical theory of rotation numbers. The *rotation number* in our context is a function $\rho\colon G \to \mathbb{R}/2\pi\mathbb{Z}$ that lifts to a quasimorphism $\widetilde{\rho}\colon \widetilde{G} \to \mathbb{R}$ of the universal cover of G. We explain the construction in the case $G = \mathrm{PSL}(2,\mathbb{R})$ and refer the reader to [BIW10, §7] for the general case. The main result is

Theorem 5.3.1 ([BIW10]). *Let $\widetilde{\phi}\colon \pi_{g,n} \to \widetilde{G}$ be a group homomorphism that covers ϕ. Then*

$$\mathrm{vol}(\phi) = -\sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right),$$

where c_i are the generators of $\pi_{g,n}$ of presentation (1.1.3).

Example 5.3.2. Let's study the case $G = \mathrm{PSL}(2,\mathbb{R})$. We fix a topological group structure on $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ by fixing a unit e in the fibre over the identity. The action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle $\mathbb{R}/2\pi\mathbb{Z}$ (see Lemma A.4) gives a group homomorphism $f\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z})$. This action lifts to a faithful action of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ on the universal cover $\mathbb{R}/2\pi\mathbb{Z}$. The classical rotation number is a function $\mathrm{rot}\colon \mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}$, see [Ghy01]. The quasimorphism $\widetilde{\rho}\colon \widetilde{\mathrm{PSL}(2,\mathbb{R})} \to \mathbb{R}$ is the unique lift of $\rho := \mathrm{rot} \circ f$ satisfying $\widetilde{\rho}(e) = 0$.

We can describe ρ more explicitly by considering conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$. Recall that, if $\mathcal{E}$ denotes the set of elliptic conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$, then there is a well-defined angle function $\vartheta\colon \mathcal{E} \to (0, 2\pi)$, see Lemma A.7. It extends to an upper semi-continuous function $\overline{\vartheta}\colon \mathrm{PSL}(2,\mathbb{R}) \to [0, 2\pi]$ by

$$\overline{\vartheta}(A) := \begin{cases} \vartheta(A), & \text{if } A \text{ is elliptic,} \\ 0, & \text{if } A \text{ is hyperbolic or positively parabolic,} \\ 2\pi, & \text{if } A \text{ is the identity or negatively parabolic.} \end{cases} \tag{5.3.1}$$

The notions of positively and negatively parabolic refer to the two conjugacy classes of parabolic elements in $\mathrm{PSL}(2,\mathbb{R})$ represented by (A.6). The definition of the function $\overline{\vartheta}$ is ad hoc, however it satisfies $\overline{\vartheta} = \rho$ modulo 2π. In particular, the correction term

$$k(\phi) := \frac{1}{2\pi}\left(\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) - \sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right)\right) \tag{5.3.2}$$

is an integer called the *relative Euler class* of ϕ. The definition of the relative Euler class very much depends on the choice of the extension $\overline{\vartheta}$ of ϑ. Theorem 5.3.1 implies

$$k(\phi) = \frac{1}{2\pi}\left(\mathrm{vol}(\phi) + \sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))\right).$$

The range of the relative Euler class over $\mathrm{Hom}(\pi_{g,n}, G)$ was studied in [DT19]. The authors proved that

Proposition 5.3.3 ([DT19]). *Let $\phi\colon \pi_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$ be a representation. Then*

$$k(\phi) \leqslant \max\left\{|\chi(\Sigma_{g,n})|, \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))\right\}.$$

Remark 5.3.4. Observe that, as soon as $g \geqslant 1$, then $|\chi(\Sigma_{g,n})| \geqslant n \geqslant \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))$ and thus the inequality $k(\phi) \leqslant |\chi(\Sigma_{g,n})|$ prevails. In the case $g = 0$, it is however possible that

$\frac{1}{2\pi}\sum_{i=1}^{n}\overline{\vartheta}(\phi(c_i)) > |\chi(\Sigma_{0,n})|$.

Chapter 6

Mapping class group dynamics

We expand on some results and remarks from Section 1.3. Let G be a Lie group and Γ be a finitely generated group. Recall that the $\mathrm{Aut}(\Gamma)$-action on the representation variety $\mathrm{Hom}(\Gamma, G)$ descends to an action of the outer automorphisms group $\mathrm{Out}(\Gamma)$ on the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. This action preserves the analytic/algebraic structure of $\mathrm{Hom}(\Gamma, G)$ by Lemma 1.3.1. When $\Gamma = \pi_{g,n}$ is a surface group, then $\mathrm{Out}(\pi_{g,n})$ contains the mapping class group of the surface $\Sigma_{g,n}$ as a subgroup, compare Example 1.3.2. The induced action is the so-called *mapping class group action on character varieties.*

We start with some general considerations on the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ and then specialize to the case of a surface group.

6.1 Remarks on the $\mathrm{Aut}(\Gamma)$-action

Lemma 6.1.1. *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves the subspaces of (very) regular, reductive, irreducible, good and (almost) Zariski dense representations.*

Proof. All these particular notions of representations are defined in terms of the image of the representation. However, for any $\tau \in \mathrm{Aut}(\Gamma)$ and $\phi \in \mathrm{Hom}(\Gamma, G)$, it holds that $\phi(\Gamma) = (\phi \circ \tau)(\Gamma)$. □

A consequence of Lemma 6.1.1 is that the $\mathrm{Out}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ restricts to an action of $\mathrm{Out}(\Gamma)$ on the GIT character variety $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$ (by Theorem 3.5.3, assuming G is a reductive complex algebraic group) and on the analytic character variety $\mathrm{Rep}^{\infty}(\pi_{g,0}, G)$.

Lemma 6.1.2. *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves closed orbits.*

Proof. This is an immediate consequence of Lemma 1.3.1. □

In particular, Lemma 6.1.2 implies that the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ induces an $\mathrm{Out}(\Gamma)$-action on the $\mathcal{T}_1$ character variety $\mathrm{Rep}^{\mathcal{T}_1}(\pi_{g,0}, G)$. It is not clear to the author whether there is an induced action of $\mathrm{Out}(\Gamma)$ on the Hausdorff character variety in general.

6.2 Generalities about mapping class groups

The *mapping class group* of a closed and oriented surface $\Sigma_{g,0}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,0}$. In the case of a punctured oriented surface $\Sigma_{g,n}$, the mapping class group is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that fix each puncture individually[1]. The mapping class group is denoted by $\mathrm{Mod}(\Sigma_{g,n})$ and the isotopy class of an orientation-preserving homeomorphism $f\colon \Sigma_{g,n} \to \Sigma_{g,n}$ is denoted $[f] \in \mathrm{Mod}(\Sigma_{g,n})$. The group law is given by composition and the identity element correspond to the identity homeomorphism.

Theorem 6.2.1. *The mapping class group is finitely presented. Generators can be chosen to be Dehn twists along simple closed curves on $\Sigma_{g,n}$.*

More details about Theorem 6.2.1, including proof and explicit generating family, can be found in [FM12, §4]. In [GW17], the question of the minimal number of generators of $\mathrm{Mod}(\Sigma_{0,n})$ is treated, see also Remark **??**.

A homeomorphism f of $\Sigma_{g,n}$ induces a group isomorphism $\pi_1(\Sigma_{g,n}, x) \to \pi_1(\Sigma_{g,n}, f(x))$. After choosing a continuous path from x to $f(x)$, we get an induced automorphism of the fundamental group of $\Sigma_{g,n}$ (that depends up to conjugation on the choice of the path). This gives a group homomorphism

$$\mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Out}(\pi_{g,n}).$$

The Dehn–Nielsen Theorem says that it is injective and provides a description of its image.

Theorem 6.2.2 (Dehn–Nielsen Theorem)**.** *The mapping class group $\mathrm{Mod}(\Sigma_{g,0})$ is an index two subgroup of $\mathrm{Out}(\pi_{g,0})$ for $g \geqslant 1$ (and is trivial for $g = 0$). Moreover, if $\Sigma_{g,n}$ has negative Euler characteristic, then the mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ is an index two subgroup of $\mathrm{Out}^\star(\pi_{g,n})$, where $\mathrm{Out}^\star(\pi_{g,n})$ is the subgroup of $\mathrm{Out}(\pi_{g,n})$ that consists of the outer automorphisms that act by conjugation on the generators c_i of $\pi_{g,n}$ (in the presentation (1.1.3)).*

We refer the reader to [FM12, §8] for more considerations on the Dehn–Nielsen Theorem. Theorem 6.2.2 implies that the $\mathrm{Aut}(\pi_{g,0})$-action on the representation variety $\mathrm{Hom}(\pi_{g,0}, G)$ induces an action

$$\mathrm{Mod}(\Sigma_{g,0}) \circlearrowright \mathrm{Hom}(\pi_{g,0}, G)/\,\mathrm{Inn}(G).$$

The action is analytic/algebraic on the regular part of the quotient by Lemma 1.3.1. In the case of a punctured surface, the action of $\mathrm{Aut}(\pi_{g,n})$ on $\mathrm{Hom}(\pi_{g,n}, G)$ restricts to an action of $\mathrm{Aut}^\star(\pi_{g,n})$ on any relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. This gives, by Theorem 6.2.2, an action

$$\mathrm{Mod}(\Sigma_{g,n}) \circlearrowright \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\,\mathrm{Inn}(G),$$

for any choice of conjugacy classes $\mathcal{C} \in (G/G)^n$. These two actions are what we call the *mapping class group action on character varieties.*

[1] In the terminology of [FM12], if punctures are fixed individually, then the group is called the *pure mapping class group*. It contrasts with the mapping class group where punctures can be permuted.

6.3 Properties of the mapping class group action

The first property is that the mapping class group action preserves the Goldman symplectic form. We start with the case of a closed surface. Let $[f] \in \mathrm{Mod}(\Sigma_{g,0})$ and take any $\tau \in \mathrm{Aut}(\pi_{g,0})$ that lies over the image of $[f]$ inside $\mathrm{Out}(\pi_{g,0})$. We choose the generator $[\pi_{g,0}]$ of $H_2(\pi_{g,0}, \mathbb{Z})$ that corresponds to the orientation of the surface $\Sigma_{g,0}$. Since f is orientation-preserving, it holds that $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$. For any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the automorphism τ induces a map $(d\tau)_\phi \colon Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \to Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau})$, $v \mapsto v \circ \tau$, on the Zariski tangent spaces to the representation variety.

Lemma 6.3.1. *If $\omega_{\mathcal{G}}$ denotes the Goldman symplectic form from Definition 4.0.2, then, for any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the following diagram commutes*

$$\begin{tikzcd}
Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \times Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \arrow[r, "(\omega_{\mathcal{G}})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_{\mathcal{G}})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

In other words, it holds that

$$\tau^*\omega_{\mathcal{G}} = \omega_{\mathcal{G}}.$$

Proof. Let $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the pairing used in the definition of $\omega_{\mathcal{G}}$. For any $v, w \in Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$, we have

$$\begin{aligned}
(\omega_{\mathcal{G}})_{\phi\circ\tau}(v \circ \tau, w \circ \tau) &= B(v \circ \tau, w \circ \tau) \frown [\pi_{g,0}] \\
&= B(v, w) \frown \tau_*[\pi_{g,0}].
\end{aligned}$$

Since $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$, we conclude $(\omega_{\mathcal{G}})_{\phi\circ\tau}(v \circ \tau, w \circ \tau) = (\omega_{\mathcal{G}})_\phi(v, w)$. ☐

As a consequence of Lemma 6.3.1, we obtain that the $\mathrm{Mod}(\Sigma_{g,0})$-action on the quotient $\mathrm{Hom}(\pi_{g,0}, G)/\mathrm{Inn}(G)$ preserves the Goldman symplectic measure $\nu_{\mathcal{G}}$ from Definition 4.2.11.

The situation is similar for punctured surfaces. Let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. The generator $[\pi_{g,n}]$ of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z})$ is again chosen to correspond to the orientation of the surface $\Sigma_{g,n}$. Similarly as before, $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$. Moreover, the map $(d\tau)_\phi$ restricts to to a map $(d\tau)_\phi \colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau})$. Indeed, note that if $v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i$ and $\tau(c_i) = g_i c_i g_i^{-1}$, then

$$(v \circ \tau)(c_i) = \big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big) - \mathrm{Ad}\left((\phi \circ \tau)(c_i)\right)\big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big).$$

Lemma 6.3.2. *If $\omega_{\mathcal{G}}$ denotes the Goldman symplectic form from Definition 4.2.9, then, for any*

$\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$, *the following diagram commutes*

$$\begin{tikzcd}
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \arrow[r, "(\omega_{\mathcal{G}})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_{\mathcal{G}})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

In other words, it holds that

$$\tau^* \omega_{\mathcal{G}} = \omega_{\mathcal{G}}.$$

The proof is analogous to the proof of Lemma 6.3.1.

The second property is that the mapping class group action also preserves the volume of a representation. As before, let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. Again, $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$.

Lemma 6.3.3. *Let G be a Hermitian Lie group. For any $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$, it holds that*

$$\mathrm{vol}(\phi \circ \tau) = \mathrm{vol}(\phi).$$

Proof. We compute directly from Definition 5.1.4 that

$$\begin{aligned}
\mathrm{vol}(\phi \circ \tau) &= j^{-1}\big((\phi \circ \tau)^* \kappa\big) \frown [\pi_{g,n}] \\
&= j^{-1}\big(\tau^* \phi^* \kappa\big) \frown [\pi_{g,n}] \\
&= j^{-1}\big(\phi^* \kappa\big) \frown \tau_*[\pi_{g,n}].
\end{aligned}$$

We conclude by using $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$. $\square$

A The groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$

This appendix is a reminder of the basic properties of the Lie groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ and of some relevant results.

A.1 The group $\mathrm{SL}(2,\mathbb{C})$

The group $\mathrm{SL}(2,\mathbb{C})$ is the group of complex 2×2 matrices with determinant 1. It is a complex algebraic group of complex dimension 3. It is also a non-compact simple Lie group. Its center is $Z(\mathrm{SL}(2,\mathbb{C})) = \{\pm I\}$. The only proper parabolic subgroup of $\mathrm{SL}(2,\mathbb{C})$, up to conjugation, is the subgroup of upper triangular matrices. We are interested in the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$ and its irreducible subgroups in the sense of Definition 2.2.7.

Theorem A.1 ([Sit75])**.** *Let G be an infinite algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then one of the following holds:*

1. $\dim G = 3$ *and* $G = \mathrm{SL}(2,\mathbb{C})$,
2. $\dim G = 2$ *and G is conjugate to the parabolic subgroup of upper triangular matrices,*
3. $\dim G = 1$*, in which case there are three possibilities*
   (a) *G is conjugate to*
   $$\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a^n = 1,\ a, b \in \mathbb{C}\right\},$$
   and G° is unipotent,
   (b) *G is conjugate to*
   $$\left\{\begin{pmatrix} a & \lambda c \\ c & a\end{pmatrix} : a^2 - \lambda c^2 = 1,\ a, c \in \mathbb{C}\right\},$$
   for some $\lambda \in \mathbb{C}^\times$, and G is connected and diagonalizable,
   (c) *G is conjugate to*
   $$\mathrm{SO}^\lambda := \left\{\begin{pmatrix} a & \lambda c \\ c & a\end{pmatrix} : a^2 - \lambda c^2 = 1,\ a, c \in \mathbb{C}\right\} \cup \left\{\begin{pmatrix} a & -\lambda c \\ c & -a\end{pmatrix} : -a^2 + \lambda c^2 = 1,\ a, c \in \mathbb{C}\right\},$$
   for some $\lambda \in \mathbb{C}^\times$, and G° is diagonalizable.

Recall that the algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$ of dimension 0 are necessarily finite (because algebraic varieties have finitely many connected components in the usual topology, as pointed out earlier). They are well-understood, see e.g. [Sit75, Prop. 1.2]. Also observe that $\mathrm{SO}(2,\mathbb{C}) = \mathrm{SO}^{-1}$ in the notation above. The irreducible subgroups of $\mathrm{SL}(2,\mathbb{C})$ fall into three categories.

Theorem A.2 ([YCo])**.** *Let G be an irreducible subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then one of the following holds:*

1. *G is Zariski dense in* $\mathrm{SL}(2,\mathbb{C})$,

2. G is finite and non-abelian,

3. the Zariski closure of G is conjugate to

$$\Delta := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} : a \in \mathbb{C}^\times \right\}.$$

Observe that the matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ conjugates Δ to SO^1 in the notation of Theorem A.1. In particular, Δ is Zariski closed. It is also disconnected and Δ° is the subgroup of diagonal matrices. The anti-diagonal matrices in Δ have order 4.

Remark A.3. It was established in Lemma 2.2.22 that Zariski dense representations into any algebraic group are irreducible. The converse statement for $\mathrm{SL}(2,\mathbb{C})$ can sometimes be found in the literature, see e.g. [Mon16, Rem. 2.13]. It is not true. For instance, given a finite non-abelian subgroup G of $\mathrm{SL}(2,\mathbb{C})$ of order g, then there is a surjective group homomorphism $F_g \to G$, where $F_g = \langle \gamma_1, \ldots, \gamma_g \rangle$ is the free group on g generators. The fundamental group of a closed surface of genus g maps surjectively to F_g by $a_i, b_i \mapsto \gamma_i$, where a_i, b_i refer to the presentation (1.1.3). This gives two irreducible representations $\pi_{g,0} \to \mathrm{SL}(2,\mathbb{C})$ and $F_g \to \mathrm{SL}(2,\mathbb{C})$ that are irreducible but not Zariski dense. It is also possible to build an irreducible representation of a closed surface group with image inside Δ.

A.2 The groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$

The group $\mathrm{SL}(2,\mathbb{R})$ is the subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of real matrices. It is a real algebraic group of real dimension 3 that has the topology of an open solid torus. It is a non-compact simple Lie group with center $Z(\mathrm{SL}(2,\mathbb{R})) = \{\pm I\}$. The center-free quotient $\mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ is denoted $\mathrm{PSL}(2,\mathbb{R})$. The group $\mathrm{SL}(2,\mathbb{R})$ is Zariski dense inside $\mathrm{SL}(2,\mathbb{C})$ (actually, even the group $\mathrm{SL}(2,\mathbb{Z})$ is Zariski dense in $\mathrm{SL}(2,\mathbb{C})$). The maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2,\mathbb{R})$. Note that $\mathrm{SO}(2,\mathbb{R})$ is Zariski closed inside $\mathrm{SL}(2,\mathbb{R})$, but the Zariski closure of $\mathrm{SO}(2,\mathbb{R})$ inside $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SO}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SU}(1,1)$. The group $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the matrix group $\mathrm{SO}(2,1)^\circ$ of special linear transformations of $\mathbb{R}^3$ preserving the Hermitian form $y^2 - xz$ via the map

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

The group $\mathrm{PSL}(2,\mathbb{R})$ can be identified with the group of orientation-preserving isometries of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. It acts on $\mathbb{H}$ by Möbius transformations

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

The action extends to the boundary $\partial\mathbb{H}$ of the upper half-plane.

Lemma A.4. *The action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\partial\mathbb{H}$ *is isomorphic to the action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$.

Proof. Identifying $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, one can define a homeomorphism $f : \partial\mathbb{H} \to \mathbb{RP}^1$ by $x \mapsto [1 : x]$ and $\infty \mapsto [0 : 1]$. We claim that f conjugates the two actions of $\mathrm{PSL}(2, \mathbb{R})$. Indeed, it is sufficient to compare stabilizers and it is easy to see that the stabilizer of $[1 : 0] \in \mathbb{RP}^1$ and that of $0 \in \partial\mathbb{H}$ coincide with the subgroup of upper triangular matrices in $\mathrm{PSL}(2, \mathbb{R})$. $\square$

The open subspace of $\mathrm{PSL}(2, \mathbb{R})$ consisting of elements whose trace in absolute value is smaller than 2 is called the subspace of *elliptic* elements of $\mathrm{PSL}(2, \mathbb{R})$. It is denoted $\mathcal{E} \subset \mathrm{PSL}(2, \mathbb{R})$. Equivalently, an element of $\mathrm{PSL}(2, \mathbb{R})$ is elliptic if and only if it has a unique fixed point in $\mathbb{H}$.

Lemma A.5. *If* $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is elliptic, then* $b \neq 0$ *and* $c \neq 0$.

Proof. If $b = 0$ or $c = 0$, then $\det(A) = ad = 1$. So, $\mathrm{Tr}(A)^2 = (a + d)^2 \geqslant 4ad = 4$ and A is not elliptic. $\square$

Let $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element of $\mathrm{PSL}(2, \mathbb{R})$. We denote the unique fixed point of A in $\mathbb{H}$ by $\mathrm{fix}(A)$. It defines a map $\mathrm{fix} : \mathcal{E} \to \mathbb{H}$.

Lemma A.6. *The unique fixed point of* A *is*

$$\mathrm{fix}(A) = \frac{a - d}{2c} + i \cdot \frac{\sqrt{4 - (a + d)^2}}{2|c|}, \tag{A.1}$$

and the map $\mathrm{fix} : \mathcal{E} \to \mathbb{H}$ *is analytic.*

Proof. The first assertion is a straightforward computation. Since $c \neq 0$ by Lemma A.5, the map $\mathrm{fix} : \mathcal{E} \to \mathbb{H}$ is analytic. $\square$

The elliptic elements of $\mathrm{PSL}(2, \mathbb{R})$ that fix the complex unit $i \in \mathbb{H}$ are of the form

$$\mathrm{rot}_\vartheta := \pm\begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \tag{A.2}$$

for $\vartheta \in (0, 2\pi)$. Every $A \in \mathcal{E}$ is conjugate to a unique $\mathrm{rot}_{\vartheta(A)}$. This defines a function $\vartheta : \mathcal{E} \to (0, 2\pi)$. The number $\vartheta(A) \in (0, 2\pi)$ is called the *angle of rotation* of A.

Lemma A.7. *The angle of rotation of* A *is*

$$\vartheta(A) = \arctan\left(\frac{-c}{|c|} \cdot \frac{a + d}{(a + d)^2 - 2}\sqrt{4 - (a + d)^2}\right) + \varepsilon(A), \tag{A.3}$$

where

$$\varepsilon(A) := \begin{cases} 0, & \text{if } (a + d)^2 > 2 \text{ and } (a + d)\frac{-c}{|c|} > 0, \\ \pi, & \text{if } (a + d)^2 < 2, \\ 2\pi, & \text{if } (a + d)^2 > 2 \text{ and } (a + d)\frac{-c}{|c|} < 0. \end{cases}$$

Moreover, the function $\vartheta : \mathcal{E} \to (0, 2\pi)$ *is analytic.*

Proof. The number $\vartheta(A)$ can be computed as the complex argument of the complex number

$$\left.\frac{dA}{dz}\right|_{z=\operatorname{fix} A} = \left(\frac{(a+d)^2}{2} - 1\right) - i \cdot (a+d)\frac{c}{|c|}\frac{\sqrt{4-(a+d)^2}}{2}. \tag{A.4}$$

Observe that the imaginary part of (A.4) vanishes if and only if $a+d = 0$, in which case its real part is equal to -1. This means that the complex number defined by (A.4) takes values inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$. If we think of the complex argument of a number inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$ as a function $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0} \to (0, 2\pi)$, then it is analytic. This shows that $\vartheta \colon \mathcal{E} \to (0, 2\pi)$ is an analytic function. □

Lemma A.8. *The map*

$$(\operatorname{fix}, \vartheta) \colon \mathcal{E} \to \mathbb{H} \times (0, 2\pi)$$

is an analytic diffeomorphism that identifies the subset of elliptic elements in $\mathrm{PSL}(2,\mathbb{R})$ *with an open ball.*

Proof. We explained above that the map $(\operatorname{fix}, \vartheta)$ is analytic. The inverse map sends a point $z = x + i \cdot y \in \mathbb{H}$ and an angle $\vartheta \in (0, 2\pi)$ to the elliptic element

$$\operatorname{rot}_\vartheta(z) = \pm \begin{pmatrix} \cos(\vartheta/2) - xy^{-1}\sin(\vartheta/2) & (x^2y^{-1} + y)\sin(\vartheta/2) \\ -y^{-1}\sin(\vartheta/2) & \cos(\vartheta/2) + xy^{-1}\sin(\vartheta/2) \end{pmatrix}. \tag{A.5}$$

Indeed, an immediate computation gives

$$\begin{aligned} \operatorname{fix}(\operatorname{rot}_\vartheta(z)) &= \frac{-2xy^{-1}\sin(\vartheta/2)}{-2y^{-1}\sin(\vartheta/2)} + i \cdot \frac{2\sin(\vartheta/2)}{2y^{-1}\sin(\vartheta/2)} \\ &= x + iy, \end{aligned}$$

and

$$\begin{aligned} \vartheta(\operatorname{rot}_\vartheta(z)) &= \arg\left(\left(\frac{4\cos(\vartheta/2)^2}{2} - 1\right) - i \cdot (2\cos(\vartheta/2)) \cdot (-1) \cdot \frac{2\sin(\vartheta/2)}{2}\right) \\ &= \arg(\cos(\vartheta) + i\sin(\vartheta)) \\ &= \vartheta. \end{aligned}$$

□

The elements of $\mathrm{PSL}(2,\mathbb{R})$ whose trace in absolute value is equal to 2 are called *parabolic*. Parabolic elements are those that have a unique fixed point of the boundary of $\mathbb{H}$. There are two conjugacy classes of parabolic elements represented by

$$\operatorname{par}^+ := \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \operatorname{par}^- := \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{A.6}$$

The elements conjugate to par^+ are called *positively parabolic* and those conjugate to par^- *negatively parabolic*. Each conjugacy class of parabolic elements is an open annulus whose closures intersect at the identity.

The elements of $\mathrm{PSL}(2,\mathbb{R})$ with a trace larger than 2 in absolute value are called *hyperbolic*. Hyperbolic elements have precisely two fixed points on the boundary of $\mathbb{H}$. Any hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ is conjugate to

$$\mathrm{hyp}_\lambda := \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for a unique $\lambda > 0$. Hyperbolic conjugacy classes are open annuli.

Elliptic, parabolic and hyperbolic conjugacy classes foliate $\mathrm{PSL}(2,\mathbb{R})$ in a way that is illustrated on Figure 6.1.

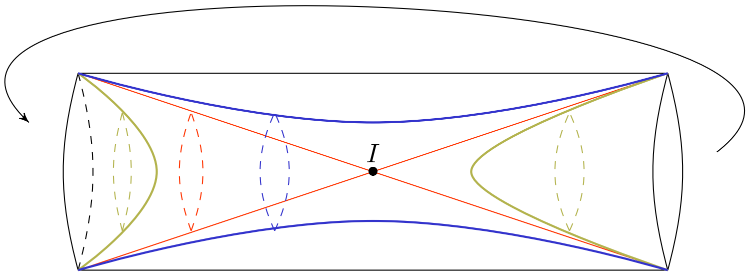


Figure 6.1: The elliptic conjugacy classes are drawn in green. They foliate an open ball into disks. The open ball is bounded by the two parabolic conjugacy classes which have the shape of two red cones joined at the identity. The hyperbolic conjugacy classes foliate an open solid torus, bounded by the red cones, into blue annuli.

The next lemma describes the centralizers of elements of $\mathrm{PSL}(2,\mathbb{R})$ according to their conjugacy class.

Lemma A.9. *The centralizers of* rot_ϑ, hyp_λ *and* par^+ *are given by*

1. $Z(\mathrm{rot}_\vartheta) = \{\mathrm{rot}_\theta : \theta \in [0, 2\pi)\} \cong \mathrm{PSO}(2,\mathbb{R})$,

2. $Z(\mathrm{hyp}_\lambda) = \{\mathrm{hyp}_t : t > 0\} \cong \mathbb{R}_{>0}$,

3. $Z(\mathrm{par}^+) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R}$.

It is worth noticing that the centralizer of an element of $\mathrm{PSL}(2,\mathbb{R})$ always consists of the identity element and of elements of the same nature (i.e. elliptic, parabolic and hyperbolic). In particular, two elements of $\mathrm{PSL}(2,\mathbb{R})$ different from the identity commute if and only if they have the same set of fixed points in $\mathbb{H} \cup \partial\mathbb{H}$.

B Group (co)homology

This appendix is a short introduction to the topics of group (co)homology and relative group (co)homology. These notions are important because group cohomology is the natural language to describe the Zariski tangent spaces to representation varieties. This note is a short summary of classical literature such as [Nos17, §7], [Löh10] and [BE78].

B.1 Definiton

We begin by recalling the definitions of group (co)homology. Group (co)homology is a functor from the category of discrete groups G with a left G–module M to the category of graded abelian groups:

$$H^*, H_* \colon \begin{pmatrix} \text{pairs of a discrete group} \\ \text{and a left module} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{graded abelian} \\ \text{groups} \end{pmatrix}.$$

By requiring G to be discrete, we obtain a topological interpretation of group (co)homology. Recall that the natural topology on the fundamental group of a space that admits a universal cover is the discrete topology, because it is the coarser topology that makes the universal cover a principal bundle for the deck transformation action. Discrete groups have the following property.

Theorem B.1 (Classifying Space Theorem)**.** *If G is a discrete group, then there is a unique connected space BG, up to canonical homotopy, called the* classifying space[2] *of G, such that*

$$\pi_1(BG) \cong G, \quad \pi_i(BG) = 0, \quad \forall i \geqslant 2.$$

A possible definition of the (co)homology of the pair (G, M), where G is a discrete group and M is a left G–module, would be to say that it is the singular (co)homology of BG with coefficients in M. We favour however a more intrinsic approach.

Let $\mathbb{Z}[G]$ be the *integral group ring* of G, i.e. the free $\mathbb{Z}$–module generated by the elements of G. Note that a G–module structure is by definition the same as a $\mathbb{Z}[G]$–module structure. Let $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ be the *augmentation map* defined by $g \mapsto 1$, $g \in G$, and extended $\mathbb{Z}$–linearly to $\mathbb{Z}[G]$. We denote by Δ the kernel of the augmentation map.

Definition B.2 (Group (co)homology)**.** The *group (co)homology* of the discrete group G with coefficients in the left G–module M is

$$H_*(G, M) := \operatorname{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M), \quad H^k(G, M) := \operatorname{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Definition B.2 uses the derived functors Tor and Ext. What this really means is that group (co)homology can be computed with projective resolutions of $\mathbb{Z}[G]$–modules. Recall that a module P is *projective* if it satisfies the following lifting property

$$\begin{tikzcd} & A \arrow[d, two heads, "\forall"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & B, \end{tikzcd}$$

[2]The names *Eilenberg-MacLane space* or $K(G,1)$*–space* are also common.

by which we mean that every morphism $P \to B$ factors through every surjective morphism $A \to B$. Equivalently, P is projective if every short exact sequence of modules

$$0 \longrightarrow A' \longrightarrow B' \xrightarrow{f} P \longrightarrow 0$$

splits, i.e. there exists a morphism of modules $h\colon P \to B'$, called *section map*, such that $f \circ h$ is the identity on P, see [Bou89, Chap. 2, §2, Prop. 4]. A *projective resolution* $\mathcal{P}$ of a module C (not necessarily projective) is an exact sequence of projective modules ending in $C \to 0$:

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} C \longrightarrow 0 \quad \text{(exact)}.$$

A projective resolution is denoted $\mathcal{P} \twoheadrightarrow C$. The fundamental property of projective resolutions is

Lemma B.3. *Any two projective resolutions of the same module are chain homotopic.*

The derived functors in Definition B.2 mean that if $\mathcal{P} \twoheadrightarrow \Delta = \operatorname{Ker}(\varepsilon)$ is the projective resolution of $\mathbb{Z}[G]$–modules

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

then

$$H_*(G; M) = H_*(\mathcal{P} \otimes_G M), \quad H^*(G; M) = H^*\big(\operatorname{Hom}_G(\mathcal{P}; M)\big).$$

In particular, $H_0(G; M) = \Delta \otimes_G M$ and the negative-degree cohomology modules vanish. Similarly, $H^0(G; M) = \operatorname{Hom}_G(\Delta; M)$. Since any two projective resolutions of Δ are chain homotopic, group (co)homology is independent of the choice of the projective resolution $\mathcal{P} \twoheadrightarrow \Delta$.

Example B.4. We compute the homology of free groups with coefficients in a trivial module M. Let $F_n = \langle a_1, \ldots, a_n \rangle$ be the free group on n elements. We claim that Δ is the free $\mathbb{Z}[F_n]$–module given by $\Delta = \langle a_1 - 1, \ldots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. The show the inclusion $\Delta \subset \langle a_1 - 1, \ldots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$, argument as follows. If $x \in \Delta$, then $x = \sum n_i h_i$ where $h_i \in F_n$ and the n_i are integers whose sum is zero. An induction on the length of h_i shows that $(h_i - 1) \in \langle a_1 - 1, \ldots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Now, since $x = \sum n_i h_i = \sum n_i (h_i - 1)$, we conclude that $x \in \langle a_1 - 1, \ldots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Since Δ is a free $\mathbb{Z}[F_n]$–module, then

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[F_n] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free, hence projective, resolution of Δ. In particular

$$H_k(F_n, M) = \begin{cases} M, & k = 0 \\ M^n, & k = 1 \\ 0, & k \geqslant 2 \end{cases}.$$

Note that this corresponds to the homology of a sphere with $n + 1$ punctures.

B.2 The bar resolution for (co)homology

Our favourite choice of projective resolution of Δ is the so-called *bar resolution*. It is defined by $P_k := \mathbb{Z}[G^{k+1}]$ for $k \geqslant 1$. Using the canonical isomorphism $M \otimes_G \mathbb{Z}[G^{k+1}] \cong M \otimes_{\mathbb{Z}} \mathbb{Z}[G^k]$, we obtain that the group homology of G with coefficients in M can be computed as the homology of the chain complex

$$C_k(G, M) := M \otimes_{\mathbb{Z}} \mathbb{Z}[G^k], \quad k \geqslant 0.$$

It is called the *bar chain complex* of G and M. The differential $\partial_k \colon C_k(G, M) \to C_{k-1}(G, M)$ is defined by

$$\begin{aligned}\partial_k(a \otimes (g_1, \ldots, g_k)) :=& g_1 \cdot a \otimes (g_2, \ldots, g_k) \\ &+ \sum_{i=1}^{k-1} (-1)^i a \otimes (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_k) \\ &+ (-1)^k a \otimes (g_1, \ldots, g_{k-1}),\end{aligned} \tag{B.1}$$

where $a \in M$ and $(g_1, \ldots, g_k) \in G^k$.

The *bar cochain complex* is given by

$$C^k(G, M) := \mathrm{Map}(G^k, M),$$

where $\mathrm{Map}(G^k; M)$ is the G–module of set-theoretic functions from G^k to M. The differential $\partial^k \colon C^{k-1}(G; M) \to C^k(G; M)$ is defined by

$$\begin{aligned}(\partial^k u)(g_1, \ldots, g_k) :=& g_1 \cdot u(g_2, \ldots, g_k) \\ &+ \sum_{i=1}^{k-1} (-1)^i u(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_k) \\ &+ (-1)^k u(g_1, \ldots, g_{k-1}),\end{aligned} \tag{B.2}$$

where $u \in \mathrm{Map}(G^{k-1}; M)$. One can easily check that the squares of the differentials ∂_k and ∂^k vanish.

There is an obvious relation between the differentials (B.1) and (B.2) given by

$$(\partial^k u)(g_1, \ldots, g_k) = \tilde{u}\big(\partial_k(1 \otimes (g_1, \ldots, g_k))\big), \tag{B.3}$$

where $\tilde{u} \colon M \otimes_{\mathbb{Z}} \mathbb{Z}[G^{k-1}] \to M$ is the unique lift of the $\mathbb{Z}$–linear map $M \times \mathbb{Z}[G^{k-1}] \to M$, $(a, (g_1, \ldots, g_k)) \mapsto a \cdot u(g_1, \ldots, g_k)$.

The sets of k-cocycles and k-coboundaries of the bar complex are denoted $Z^k(G, M)$ and $B^k(G, M)$, respectively. In particular, the 1-cocycles are

$$Z^1(G, M) := \{u \colon G \to M : u(g_1 g_2) = u(g_1) + g_1 \cdot u(g_2), \quad \forall g_1, g_2 \in G\}$$

and the 1-coboundaries are

$$B^1(G, M) := \{u \colon G \to M : \exists a \in M, \quad u(g) = g \cdot a - a, \quad \forall g \in G\}.$$

B.3 Relative group (co)homology

Let $\mathcal{K} = \{K_i : i \in I\}$ be a family of subgroups of G stable under conjugation. We define the group (co)homology of G relative to $\mathcal{K}$ with coefficients in M. Let $\mathbb{Z}[G/\mathcal{K}] := \bigoplus_{i\in I} \mathbb{Z}[G/K_i]$ be the direct sum of the free groups generated by the left cosets of K_i in G. We denote by Δ the kernel of the augmentation map $\varepsilon\colon \mathbb{Z}[G/\mathcal{K}] \to \mathbb{Z}$.

Definition B.5 (Relative group (co)homology)**.** The *relative (co)homology groups* of G relative to $\mathcal{K}$ with coefficients in the G–module M are defined by

$$H_*(G,\mathcal{K},M) := \mathrm{Tor}_{*-1}^{\mathbb{Z}[G]}(\mathbb{Z},\Delta\otimes_G M),$$

$$H^*(G,\mathcal{K},M) := \mathrm{Ext}^{*+1}_{\mathbb{Z}[G]}(\mathbb{Z},\mathrm{Hom}_G(\Delta;M)).$$

Observe that

$$H_*(G,\mathcal{K},M) = H_{*-1}(G,\Delta\otimes_G M), \tag{B.4}$$

$$H^*(G,\mathcal{K},M) = H^{*-1}(G,\mathrm{Hom}_G(\Delta;M)). \tag{B.5}$$

In particular, $H_0(G,\mathcal{K},M) = H^0(G,\mathcal{K},M) = 0$, $H_1(G,\mathcal{K},M) = \Delta\otimes_G M$ and $H^1(G,\mathcal{K},M) = \mathrm{Hom}_G(\Delta;M)$.

Remark B.6. Definition B.5 makes perfect sense even if $\mathcal{K}$ is not assumed to be closed under conjugation. This gives a notion of group (co)homology relative to any family of subgroups. However, this notion is equivalent to the former in the following sense. If $\overline{\mathcal{K}}$ denote the *conjugation closure* of $\mathcal{K}$:

$$\overline{\mathcal{K}} := \{gKg^{-1} : g\in G, K\in\mathcal{K}\},$$

then there are canonical isomorphisms

$$H_*(G,\mathcal{K},M) \cong H_*(G,\overline{\mathcal{K}},M), \quad H^*(G,\mathcal{K},M) \cong H^*(G,\overline{\mathcal{K}},M). \tag{B.6}$$

Indeed, choose a set of coset representatives $\mathcal{X}$ for $G/\mathcal{K}$. This gives an identification $\mathbb{Z}[G/\mathcal{K}] \cong \mathbb{Z}[G/\overline{\mathcal{K}}]$ which induces the desired isomorphisms. The resulting isomorphisms (B.6) are independent of the choice of $\mathcal{X}$, see [BE78, Proposition 7.5].

B.4 Bar resolution for relative (co)homology

The bar resolution for relative group (co)homology is obtained from the bar resolution for group (co)homology using the cone construction. Recall that if A and B are chain complexes and $f\colon B\to A$ is a morphism of chain complexes, then the *cone* of f is the chain complex $C(f)$ with differential d given by

$$C(f)_k := A_k\oplus B_{k-1}, \quad d(\alpha,\beta) := (-d\alpha + f(\beta), d\beta).$$

This construction produces an exact triangle of complexes $B\to A\to C(f)\to B[-1]$ where $B[-1]$ is the shifted complex obtained from B, also called the *suspension* of B. The exact triangle induces a long exact sequence in (co)homology.

We adopt the shorthand notation

$$C_k(\mathcal{K}, M) := \bigoplus_{i\in I} C_k(K_i, M), \quad C^k(\mathcal{K}, M) := \prod_{i\in I} C^k(K_i, M).$$

The *relative bar chain complex* is given by the cone of the inclusion $K_i \subset G$, i.e.

$$\begin{aligned} C_k(G, \mathcal{K}, M) : &= C_k(G, M) \oplus C_{k-1}(\mathcal{K}, M), \\ &\cong M \otimes_G \big(\mathbb{Z}[G^k] \oplus \mathbb{Z}[\mathcal{K}^{k-1}]\big). \end{aligned}$$

with differential $\partial_k \colon C_k(G, \mathcal{K}, M) \to C_{k-1}(G, \mathcal{K}, M)$ defined by

$$\partial_k(g, h) := \big(- \partial_k g + \sum_{i\in I} \imath_i h_i,\, \partial_{k-1} h\big), \tag{B.7}$$

where $g \in C_k(G; M)$ and $h = (h_i)_{i\in I} \in C_{k-1}(\mathcal{K}; M)$. Recall that at most finitely many h_i are nonzero so that the sum in (B.7) makes sense. The *relative bar cochain complex* is defined by

$$\begin{aligned} C^k(G, \mathcal{K}, M) : &= C^k(G, M) \oplus C^{k-1}(\mathcal{K}, M), \\ &\cong \mathrm{Map}\big(\mathbb{Z}[G^k] \oplus \mathbb{Z}[\mathcal{K}^{k-1}], M\big). \end{aligned}$$

The differential $\partial^k \colon C^k(G, \mathcal{K}, M) \to C^{k+1}(G, \mathcal{K}, M)$ is given by

$$\begin{aligned} \partial^k(u, f) : &= \big(\partial^k u,\, u\,\imath_i - \partial^{k-1} f_i\big) \\ &= \big(u\,\partial_{k+1},\, u\,\imath_i - f_i \partial_k\big), \end{aligned} \tag{B.8}$$

where $u \in C^k(G, M)$ and $f = (f_i)_{i\in I} \in C^{k-1}(\mathcal{K}, M)$. The second equality in (B.8) follows from the relation (B.3) which implies $u\partial_{k+1} = \partial^k u$ and $f\partial_k = \partial^{k-1} f$.

There are long exact sequences in group homology and cohomology that read

$$\ldots \longrightarrow H_k(\mathcal{K}, M) \xrightarrow{\oplus(\imath_i)_\star} H_k(G, M) \xrightarrow{j} H_k(G, \mathcal{K}, M) \xrightarrow{r} H_{k-1}(\mathcal{K}, M) \longrightarrow \ldots \tag{B.9}$$

$$\ldots \longrightarrow H^{k-1}(\mathcal{K}, M) \xrightarrow{r} H^k(G, \mathcal{K}, M) \xrightarrow{j} H^k(G, M) \xrightarrow{\times(\imath_i)^\star} H^k(\mathcal{K}, M) \longrightarrow \ldots \tag{B.10}$$

We used the shorthand notations $H_k(\mathcal{K}, M) := \bigoplus_{i\in I} H_k(K_i, M)$ and $H^k(\mathcal{K}, M) := \prod_{i\in I} H^k(K_i, M)$. The morphisms j and r are induced from the inclusion and restriction on the (co)chain complex level. The long exact sequences are obtained by applying the derived functors $\mathrm{Ext}^*_{\mathbb{Z}[G]}(\cdot, M)$ and $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\cdot, M)$ to the short exact sequence

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[G/\mathcal{K}] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

B.5 Relation to singular (co)homology

The purpose of this section is to explain how the singular (co)homology of a space relates to the group (co)homology of its fundamental group.

Definition B.7 (Eilenberg-MacLane pair)**.** A pair of topological spaces (X, Y), $Y \subset X$, is an

Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$, if X is a $K(G, 1)$–CW-complex and if $Y = \sqcup Y_i$ where each Y_i is a $K(K_i, 1)$–subcomplex of X.

Equivalently, (X, Y) is an Eilenberg-MacLane pair if each inclusion $Y_i \hookrightarrow X$ induces an injective homomorphism $\pi_1(Y_i, y_i) \hookrightarrow \pi_1(X, y_i)$ and if there exists an isomorphism $\varphi\colon \pi_1(X, y_i) \to G$ induced by a suitable choice of path connecting base points such that $\varphi(\pi_1(Y_i, y_i)) = K_i$

$$\begin{tikzcd}
\pi_1(Y_i, y_i) \arrow[r, hook] \arrow[d, "\varphi"] & \pi_1(X, y_i) \arrow[d, "\varphi"] \\
K_i \arrow[r, hook, "\imath_i"] & G.
\end{tikzcd}$$

The standard examples of Eilenberg-MacLane pairs are pairs (X, Y) where X is a $K(G, 1)$-space and Y is the boundary of X.

Theorem B.8 ([BE78])**.** *Let (X, Y) be an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. Then there exist isomorphisms in (co)homology in every degree that relates the long exact sequences of the pairs (X, Y) and $(G, \mathcal{K})$ such that the following diagram commutes (up to a minus sign for the middle square)*

$$\begin{tikzcd}
H^{k-1}(\mathcal{K}, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(G, \mathcal{K}, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(G, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(\mathcal{K}, M) \arrow[d, dashed, "\cong"] \\
H^{k-1}(Y, M) \arrow[r] & H^k(X, Y, M) \arrow[r] & H^k(X, M) \arrow[r] & H^k(Y, M).
\end{tikzcd}$$

Remark B.9. Observe that if (X, Y) is an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$, then it is also an Eilenberg-MacLane pair of type $K(G, \mathcal{K}', 1)$ where $\mathcal{K}'$ is obtained from $\mathcal{K}$ by individually conjugating its elements. So, as a byproduct of Theorem B.8, we get a natural isomorphism between the (co)homology of the pairs $(G, \mathcal{K})$ and $(G, \mathcal{K}')$. This isomorphism corresponds to the one induced by (B.6). In addition there are natural isomorphisms

$$H_\star(X, Y, M) \cong H_\star(G, \overline{\mathcal{K}}, M), \quad H^\star(X, Y, M) \cong H^\star(G, \overline{\mathcal{K}}, M),$$

where $\overline{\mathcal{K}}$ denotes the conjugation closure of $\mathcal{K}$ introduced in Remark B.6.

We refer the reader to [BE78, Thm. 1.3] for a proof of Theorem B.8.

B.6 Cup product

We introduce the cup product in group cohomology using the bar cochain complex as in [Nos17, §7]. Let G be a group and M, M' be two G–modules. Let $u \in C^k(G, M)$ and $v \in C^l(G, M')$. The *cup product* of u and v is defined as the cochain $u \smile v \in C^{k+l}(G, M \otimes_G M')$ defined by

$$u \smile v(g_1, \ldots, g_{k+l}) := u(g_1, \ldots, g_k) \otimes g_1 \ldots g_k \cdot v(g_{k+1}, \ldots, g_l). \tag{B.11}$$

Lemma B.10. *The cup product satisfies the Leibniz rule:*

$$\partial^{k+l+1}(u \smile v) = \partial^{k+1} u \smile v + (-1)^k u \smile \partial^{l+1} v.$$

The Leibniz rule implies that the cup product descends to a well-defined G-invariant product on cohomology:

$$\smile : H^k(G, M) \otimes_G H^l(G, M') \to H^{k+l}(G, M \otimes_G M').$$

Lemma B.11. *Up to the natural identification $M \otimes_G M' \cong M' \otimes_G M$, it holds that*

$$[u \smile v] = (-1)^{kl}[v \smile u], \quad \forall u \in Z^k(G, M), \forall v \in Z^l(G, M').$$

Proof. We treat the case $k = l = 1$. The other cases are similar. We start by computing the differential of $u \otimes v$ using (B.2)

$$\begin{aligned} -\partial^2(u \otimes v)(x, y) &= -u(x) \otimes v(x) + u(xy) \otimes v(xy) - x \cdot (u(y) \otimes v(y)) \\ &= u(x) \otimes x \cdot u(y) + x \cdot u(y) \otimes v(x) \\ &= u \smile v(x, y) + v \smile u(x, y), \end{aligned}$$

where in the second equality we used the cocycle property $u(xy) = u(x) + x \cdot u(y)$. This shows that $u \smile v + v \smile u$ is a coboundary. □

The cup product can be defined on relative cohomology as follows. Let $u \in C^k(G, M)$ and $f \in C^{k-1}(\mathcal{K}, M)$, and $v \in C^l(G, M')$. Define the cup product of (u, f) with v to be the cochain

$$(u \smile v, f \smile v) \in C^{k+l}(G, \mathcal{K}, M \otimes_G M').$$

It induces a cup product in relative cohomology

$$\smile : H^k(G, \mathcal{K}, M) \otimes_G H^l(G, M') \to H^{k+l}(G, \mathcal{K}, M \otimes_G M'). \tag{B.12}$$

B.7 Cap product and Poincaré duality

The purpose of [BE78] was to describe a notion of Poincaré duality for group pairs. This can be done as follows.

Let $\mathcal{P} \twoheadrightarrow \mathbb{Z}$ be a projective resolution of G–modules. Then $\mathcal{P} \otimes_G \mathcal{P}$ is a projective resolution of $\mathbb{Z}$ for the diagonal G–action on $\mathcal{P} \otimes_G \mathcal{P}$. Let $g = p \otimes q \otimes a \in (\mathcal{P} \otimes_G \mathcal{P}) \otimes_G M$ and $u \in \mathrm{Hom}_G(\mathcal{P}, M')$. The *cap product* of g and u is defined to be

$$g \frown u := q \otimes (a \otimes u(p)) \in P \otimes_G (M \otimes_G M').$$

Lemma B.12. *The cap product is a well-defined operation on complexes and satisfies the Leibniz rule*

$$\partial_k(g \frown u) = (-1)^l \partial_{k+l}\, g \frown u + g \frown \partial^l u.$$

The induced cap product on (co)homology is

$$\frown : H_{k+l}(G, M) \otimes_G H^k(G, M') \to H_l(G, M \otimes_G M')$$

The definition of the cap product in relative (co)homology uses the pairing

$$\begin{aligned} B\colon (\Delta \otimes_G M) \otimes_G \operatorname{Hom}_G(\Delta, M') &\to M \otimes_G M' \\ (g \otimes a) \otimes u &\mapsto a \otimes u(g). \end{aligned} \tag{B.13}$$

The *cap product on relative group (co)homology* is the dashed arrow that makes the following diagram commute

$$\begin{tikzcd} H_{k+l}(G, \mathcal{K}, M) \otimes_G H^l(G, \mathcal{K}, M') \arrow[r, dashed, "\frown"] \arrow[d, equal] & H_l(G, M \otimes_G M') \\ H_{k+l-1}(G, \Delta \otimes_G M) \otimes_G H^{k-1}(G, \operatorname{Hom}_G(\Delta, M')) \arrow[ur, bend right, "B_* \frown"'] & \end{tikzcd}$$

The equality in the first column is an application of (B.4) and (B.5).

Using a modified version of the pairing (B.13), one can define a second variant of the cap product

$$\frown\colon H_{k+l}(G, \mathcal{K}, M) \otimes_G H^k(G, M') \to H_l(G, \mathcal{K}, M \otimes_G M').$$

The two versions of the cup product are natural operations in group (co)homology, see [BE78] for more details.

The cap product maps the long exact sequence in cohomology for the pair $(G, \mathcal{K})$ to its long exact sequence in homology. This commutes with the corresponding map in singular homology under the isomorphism of Theorem B.8. Indeed, let (X, Y) denote an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. For any $e \in H_n(G, \mathcal{K}, M)$, let $\overline{e} \in H_n(X, Y; M)$ be the image of e under the isomorphism of Theorem B.8. The following diagram commutes for $k = 0, \ldots, n$ (up to some minus signs depending on the degree of the two lower squares, see [BE78] for complete details)

$$\begin{tikzcd} H_{n-k}(G, \mathcal{K}, M \otimes_G M') \arrow[r] & H_{n-k-1}(\mathcal{K}, M \otimes_G M') \arrow[r] & H_{n-k-1}(G, M \otimes_G M') \\ H^k(G, M') \arrow[r] \arrow[u, "e\frown"] \arrow[d, "\cong"] & H^k(\mathcal{K}, M') \arrow[r] \arrow[u, "r(e)\frown"] \arrow[d, "\cong"] & H^{k+1}(G, \mathcal{K}, M') \arrow[u, "e\frown"] \arrow[d, "\cong"] \\ H^k(X, M') \arrow[r] \arrow[d, "\overline{e}\frown"] & H^k(Y, M') \arrow[r] \arrow[d, "r(\overline{e})\frown"] & H^{k+1}(X, Y, M') \arrow[d, "\overline{e}\frown"] \\ H_{n-k}(X, Y, M \otimes_G M') \arrow[r] & H_{n-k-1}(Y, M \otimes_G M') \arrow[r] & H_{n-k-1}(X, M \otimes_G M'). \end{tikzcd}$$

Here, r denotes the connecting morphism of the long exact sequence (B.9). In particular, the following square commutes

$$\begin{tikzcd} H^k(X, Y, M') \arrow[d, "\overline{e}\frown"] & H^k(G, \mathcal{K}, M') \arrow[l, "\cong"'] \arrow[d, "e\frown"] \\ H_{n-k}(X, M \otimes_G M') & H_{n-k}(G, M \otimes_G M'). \arrow[l, "\cong"'] \end{tikzcd}$$

Poincaré duality for de Rham cohomology says that if X is a smooth, compact, connected manifold of dimension n, and $[X]$ is a generator of $H_n(X; \mathbb{Z}) \cong \mathbb{Z}$, then the cap product with $[X]$ is an isomorphism

$$[X] \frown\colon H^k_{dR}(X, \mathbb{R}) \xrightarrow{\cong} H_{n-k}(X, \mathbb{R}), \quad k = 0, \ldots, n.$$

In the context of group (co)homology, one introduces the notion of Poincaré duality pairs.

Definition B.13 ((Poincaré) duality pairs)**.** The pair $(G, \mathcal{K})$ is called a *duality pair of dimension* n, in short a D^n*-pair*, if there exists a G–module N and an element $e \in H_n(G, \mathcal{K}, N)$ such that both

- $e \frown : H^k(G, M) \to H_{n-k}(G, \mathcal{K}, N \otimes_G M)$,
- $e \frown : H^k(G, \mathcal{K}, M) \to H_{n-k}(G, N \otimes_G M)$

are isomorphisms for every $k = 0, \ldots, n$ and for every G–module M. Moreover, if N can be chosen to be isomorphic to $\mathbb{Z}$ as a group, then $(G, \mathcal{K})$ is called a *Poincaré duality pair of dimension* n, in short a PD^n*-pair*.

If $(G, \mathcal{K})$ is a D^n-pair, then by letting $M = \mathbb{Z}[G]$ and $k = n$, we obtain $H^n(G, \mathcal{K}, \mathbb{Z}[G]) \cong H_0(G, N \otimes_G \mathbb{Z}[G]) \cong N$. Therefore, a duality pair determines a unique *dualizing module* N up to isomorphism. For a PD^n-pair we call each of the two generators of $H_n(G, \mathcal{K}, N) \cong \mathbb{Z}$ a *fundamental class* of $(G, \mathcal{K})$.

Example B.14. Let X be a smooth, compact, connected, manifold of dimension n with non-empty boundary ∂X. Let $[X, \partial X] \in H_n(X, \partial X, \mathbb{Z})$ be a fundamental class. Assume that $(X, \partial X)$ an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. Then $(G, \mathcal{K})$ is a PD^n-pair with fundamental class $[G, \mathcal{K}]$ given by the image of $[X, \partial X]$ under the isomorphism of Theorem B.8. In particular, the following diagram commutes

$$\begin{tikzcd}
H^n_{dR}(X, \partial X, \mathbb{R}) \arrow[r, "{[X,\partial X]\frown}"] & H_0(X, \mathbb{R}) \\
H^n(G, \mathcal{K}, \mathbb{R}) \arrow[u, "\cong"] \arrow[r, "{[G,\mathcal{K}]\frown}"] & H_0(G, \mathbb{R}). \arrow[u, "\cong"]
\end{tikzcd}$$

Here, $\mathbb{R}$ is the trivial G–module.

Observe that if $(G, \mathcal{K})$ is a D^n-pair, then there exists an induced isomorphism

$$r(e) \frown : \prod_{i \in I} H^k(K_i; M') \to \bigoplus_{i \in I} H_{n-k-1}(K_i; M \otimes_G M')$$

in every degree k and for every G–modules M, M'. Therefore, $\mathcal{K}$ must be a *finite* collection of subgroups.

Lemma B.15. *Let $(G, \mathcal{K})$ be a PD^n-pair and $\mathbb{R}$ be the trivial G–module. The cap product in degree n for the bar resolution is*

$$\begin{aligned} \frown : H_n(G, \mathcal{K}, \mathbb{R}) \otimes_G H^n(G, \mathcal{K}, \mathbb{R}) &\to \mathbb{R} \\ [(g, h_1, \ldots, h_m)] \otimes [(u, f_1, \ldots, f_m)] &\mapsto u(g) - \sum_{i=1}^m f_i(h_i), \end{aligned} \tag{B.14}$$

where $u \colon G^n \to \mathbb{R}$ and $f_i \colon K_i^{n-1} \to \mathbb{R}$ have been extended $\mathbb{Z}$-linearly to $\mathbb{Z}[G^n]$, respectively $\mathbb{Z}[K_i^{n-1}]$.

Proof. We only check that (B.14) vanishes if $(g, h_1, \ldots, h_m)$ is exact. A complete proof is given in [KM96, Proposition 5.8].

The condition $\partial^n(u, f_1, \ldots, f_m) = 0$ as defined in (B.8) means that $\partial^n u = 0$ and $u\restriction_{K_i} - \partial^{n-1} f_i = 0$ for all i. Since $(g, h_1, \ldots, h_m)$ is assumed to be exact, there exist $(g', h'_1, \ldots, h'_m) \in C_{n+1}(G, \mathcal{K}, \mathbb{R})$ such that

$$\begin{aligned}(g, h_1, \ldots, h_m) &= \partial_{n+1}(g', h'_1, \ldots, h'_m) \\ &= \left(\sum_{i=1}^{m} h'_i - \partial_{n+1} g',\, \partial_n h'_1, \ldots, \partial_n h'_m\right).\end{aligned}$$

We compute

$$\begin{aligned}u(g) - \sum_{i=1}^{m} f_i(h_i) &= \sum_{i=1}^{m} u\restriction_{K_i}(h'_i) - u(\partial_{n+1} g') - \sum_{i=1}^{m} f_i(\partial_n h'_i) \\ &= \sum_{i=1}^{m} u\restriction_{K_i}(h'_i) - \partial^n u(g') - \sum_{i=1}^{m} \partial^{n-1} f_i(h'_i),\end{aligned}$$

where in the second equality we applied the relation (B.3). The last expression vanishes because $(u, f_1, \ldots, f_m)$ is closed. $\square$

B.8 Parabolic group cohomology

Parabolic group cohomology was introduced in the sixties by André Weil. We give a succinct introduction inspired from [GHJW97].

Let G be a discrete group and $\mathcal{K} = \{K_i : i \in I\}$ be a family of subgroups of G. Let M be a G–module and $k \geqslant 0$ an integer. Define the set of *parabolic cocycles* in the bar complex to be the set k-cocycle $f\colon G^k \to M$ such that all the restrictions $f\restriction_{K_i}$ are exact, i.e. belong to $B^k(K_i, M)$. The set of parabolic cocycles in degree k is denoted

$$Z^k_{par}(G, M) \subset Z^k(G, M).$$

Parabolic cocycles are thus cocycles that are exact on the boundary.

Definition B.16 (Parabolic group cohomology)**.** The *parabolic group cohomology* of G with coefficients in the G–module M is defined to be

$$H^*_{par}(G, M) := Z^*_{par}(G, M)/B^*(G, M) \subset H^*(G; M).$$

It follows from Definition B.16 that parabolic group cohomology is related to relative group cohomology as follows.

Lemma B.17. *Let $j\colon H^k(G, \mathcal{K}, M) \to H^k(G, M)$ be the morphism of the long exact sequence (B.10) for the pair $(G, \mathcal{K})$. Then,*

$$H^k_{par}(G, M) = j\big(H^k(G, \mathcal{K}, M)\big) \cong H^k(G, \mathcal{K}, M)/\operatorname{Ker}(j).$$

The Leibniz rule of Lemma B.10 implies that the kernel and the image of j are orthogonal for the cup product (B.12). In particular, there is a non-degenerate induced product

$$\smile : H^k_{par}(G, M) \otimes_G H^l_{par}(G, M') \to H^{k+l}(G, \mathcal{K}, M \otimes_G M'). \tag{B.15}$$

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